

# THE DEPTH OF A KNOT TUNNEL

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**ABSTRACT.** The theory of tunnel number 1 knots detailed in [4] provides a non-negative integer invariant  $\text{depth}(\tau)$  for a knot tunnel  $\tau$ . We give various results related to the depth invariant. Noting that it equals the minimal number of Goda-Scharlemann-Thompson “tunnel moves” [6] needed to construct the tunnel, we calculate the number of distinct minimal sequences of tunnel moves that can produce a given tunnel. Next, we give a recursion that tells the minimum bridge number of a knot having a tunnel of depth  $d$ . The growth of this value is proportional to  $(1 + \sqrt{2})^d$ , which improves known estimates of the rate of growth of bridge number as a function of the Hempel distance of the associated Heegaard splitting. We also give various upper bounds for bridge number in terms of the cabling constructions needed to produce a tunnel of a knot, showing in particular that the maximum bridge number of a knot produced by  $n$  cabling constructions is the  $(n + 2)^{n^d}$  Fibonacci number. Finally, we explicitly compute the slope parameters for the regular (or “short”) tunnels of torus knots, and find a sequence of them for which the bridge numbers of the associated knots achieve the growth rate  $(1 + \sqrt{2})^d$ .

## INTRODUCTION

This work concerns a new invariant of knot tunnels, called the *depth*. It is based on the theory of knot tunnels developed in our earlier work [4], which provides a simplicial complex  $\mathcal{D}(H)/\Gamma$  whose vertices correspond to the (equivalence classes of) tunnels of all tunnel number 1 knots. The depth invariant of a tunnel is defined to be the simplicial distance in the 1-skeleton of  $\mathcal{D}(H)/\Gamma$  from the vertex corresponding to the tunnel to the vertex corresponding to the unique tunnel of the trivial knot. In particular, the trivial tunnel is the only tunnel of depth 0. A tunnel has depth 1 exactly when it is a  $(1, 1)$ -tunnel of a  $(1, 1)$ -knot.

We denote the depth of a tunnel  $\tau$  by  $\text{depth}(\tau)$ . It is somewhat similar to the (*Hempel distance*  $\text{dist}(\tau)$ ) (see J. Johnson [7] and Y. Minsky, Y. Moriah, and S. Schleimer [8]), but is very easy to calculate in terms of the parameter description of tunnels given in [4]. The two invariants are related by the

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inequality

$$\text{dist}(\tau) - 1 \leq \text{depth}(\tau) ,$$

but the depth can be much larger than the distance. Indeed, we will see that the regular tunnels of torus knots have distance 2, but their depths can be arbitrarily large.

The depth invariant has a geometric interpretation in terms of a construction that first appeared in a paper of H. Goda, M. Scharlemann, and A. Thompson [6]. Their construction, which we call a giant step, takes a tunnel and produces a new tunnel (usually of a different knot). They proved that every tunnel could be produced starting from the tunnel of the trivial knot and applying a sequence of giant steps, and we will see from the definitions that  $\text{depth}(\tau)$  is the minimum length of a such a sequence. Unlike the construction of a knot tunnel using cabling operations, developed in [4] and reviewed in section 3 below, the choice of giant steps is usually not unique, even when one restricts to minimal sequences. Using the simplicial structure of  $\mathcal{D}(H)/\Gamma$ , we will give an algorithm to calculate the number of distinct minimal sequences of giant steps that produce a given tunnel. In particular, this provides arbitrarily complicated examples of tunnels for which the minimal giant steps sequence *is* unique, while showing that such tunnels are sparse among the set of all tunnels. The algorithm is quite effective. We have implemented it computationally [5] to find the number of distinct minimal sequences producing a tunnel, given its parameter description from [4].

We next turn to an examination of the bridge number of a tunnel number 1 knot. The first main result, theorem 7.1, gives a lower bound for the bridge number of a specific tunnel number 1 knot in terms of the parameters of a tunnel of the knot. The algorithm to compute that bound is easy, and we have implemented it computationally [5]. The proof of theorem 7.1 is quite easy for us since the necessary ideas and hard geometric work were already developed by Goda, Scharlemann, and Thompson [6] and Scharlemann and Thompson [11]. Theorem 7.1, together with a geometric construction involving the cabling construction of tunnels from [4], gives a general and sharp lower bound for the bridge number in terms of the depth of a tunnel:

**Minimum Bridge Number Theorem 7.5.** *For  $d \geq 1$ , the minimum bridge number of a knot having a tunnel of depth  $d$  is  $a_d$ , where  $a_1 = 2$ ,  $a_2 = 4$ , and  $a_d = 2a_{d-1} + a_{d-2}$  for  $d \geq 3$ .*

As a matrix, the recursion in theorem 7.5 is

$$\begin{pmatrix} a_{d+1} \\ a_d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_d \\ a_{d-1} \end{pmatrix} .$$

The eigenvalues of this matrix are  $1 \pm \sqrt{2}$ , showing that the asymptotic growth rate of the bridge numbers of any sequence of tunnel number 1 knots as a function of depth is at least a constant multiple of  $(1 + \sqrt{2})^d$ . This improves Lemma 2 of [7], which is that bridge number grows linearly

with distance. Since each of the giant steps of a minimal sequence increases the depth by 1, corollary 7.4 also improves Proposition 1.11 of [6], which implies that bridge number grows asymptotically at least as fast as  $2^d$ .

Of course, the Minimum Bridge Number Theorem also shows that the bound for growth rate of  $(1 + \sqrt{2})^d$  is best possible, indeed its proof tells exactly how to construct a tunnel of depth  $d$  having bridge number  $a_d$ . We will show that this growth rate is also achieved by a sequence of torus knot tunnels (each obtained by applying a cabling operation to the previous one) given in section 9. In fact, the bridge numbers of the knots of that sequence are given by the recursion in the Minimum Bridge Number Theorem, except that one starts with  $a_1 = 2$  and  $a_2 = 5$ . The terms  $a_d$  are then the minimal bridge numbers of any torus knot having a tunnel of the corresponding depth.

Besides the specific examples of torus knot tunnels we have already mentioned, we give a general algorithm to compute the slope parameters for the regular tunnel (sometimes called the “short” tunnel) of a  $(p, q)$  torus knot. The algorithm uses the continued fraction expansion of  $p/q$ . It is very effective and has been implemented computationally [5].

A more general version of the geometric construction used in proving the Minimum Bridge Number Theorem allows us to show that in general, cabling operations can be carried out rather efficiently with respect to bridge number. This leads to various upper bounds for the bridge number of a knot in terms of its tunnels. In particular,

**Theorem 8.9.** *Let  $(F_1, F_2, \dots) = (1, 1, 2, 3, \dots)$  be the Fibonacci sequence. Suppose that  $\tau$  is a regular tunnel produced by  $n$  cabling operations, of which the first  $m$  produce semisimple tunnels. Then  $\text{br}(K_{\tau_n}) \leq mF_{n-m+2} + F_{n-m+1}$ .*

For fixed  $n$ , the largest value for the upper bound in theorem 8.9 occurs when  $m = 2$ , and we show that it is sharp for this case:

**Theorem 8.10.** *The maximum bridge number of any tunnel number 1-knot having a tunnel produced by  $n$  cabling operations is  $F_{n+2}$ .*

Here is an outline of the sections of the paper. The first three sections constitute a concise review of material from [4] that we will need for the present applications. Section 4 introduces the distance and depth invariants, and gives a few results that follow quickly from [4] and work of other authors. The main applications of the paper may be then read independently. The giant steps discussed above are introduced in section 5, and the analysis of minimal sequences of giant steps is carried out in section 6. Lower bounds for bridge number, in particular the Minimum Bridge Number Theorem, are given in section 7, while section 8 has the results on upper bounds. The torus knot examples are worked out in section 9. The final section of the paper reviews how the general theory can be adapted to include tunnel number 1

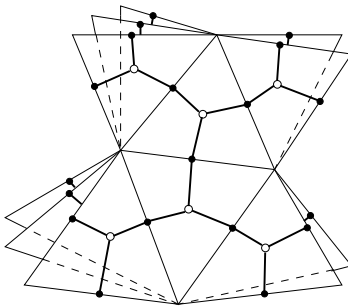


FIGURE 1. A portion of the nonseparating disk complex  $\mathcal{D}(H)$  and the tree  $\tilde{\mathcal{T}}$ . Countably many 2-simplices meet along each edge.

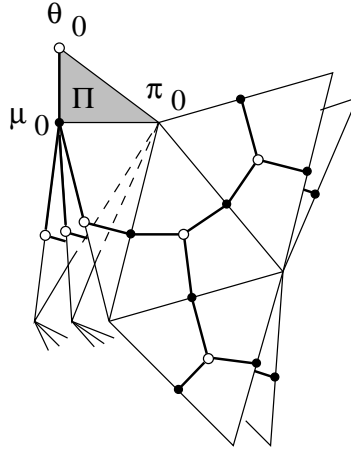
links, and indicates how the applications in the present paper extend to that case.

### 1. THE DISK COMPLEX OF AN IRREDUCIBLE 3-MANIFOLD

Let  $H$  be a genus 2 orientable handlebody, regarded as the standard unknotted handlebody in  $S^3$ . For us, a *disk in  $H$*  means a properly imbedded disk in  $H$ , which is assumed to be nonseparating unless otherwise stated. The *disk complex*  $\mathcal{D}(H)$  is a 2-dimensional, contractible simplicial complex, whose vertices are the (proper) isotopy classes of essential properly imbedded disks in  $H$ , such that a collection of  $k + 1$  vertices spans a  $k$ -simplex if and only if they admit a set of pairwise-disjoint representatives. Each 1-simplex of  $\mathcal{D}(H)$  is a face of countably many 2-simplices. As suggested by figure 1,  $\mathcal{D}(H)$  grows outward from any of its 2-simplices in a treelike way. In fact, it deformation retracts to the tree  $\tilde{\mathcal{T}}$  seen in figure 1.

Each tunnel of a tunnel number 1 knot determines a collection of disks in  $H$  as follows. The tunnel is a 1-handle attached to a regular neighborhood of the knot to form an unknotted genus-2 handlebody. An isotopy carrying this handlebody to  $H$  carries a cocore 2-disk of that 1-handle to a nonseparating disk in  $H$ , and carries the tunnel number 1 knot to a core circle of the solid torus obtained by cutting  $H$  along the image disk in  $H$ . The indeterminacy of this isotopy is the group of isotopy classes of orientation-preserving homeomorphisms of  $S^3$  that preserve  $H$ . This group is called the *Goeritz group*  $\mathcal{G}$ . Work of M. Scharlemann [10] and E. Akbas [1] proves that  $\mathcal{G}$  is finitely presented, and even provides a simple presentation of it.

Since two disks in  $H$  determine equivalent tunnels exactly when they differ by an isotopy moving  $H$  through  $S^3$ , the collection of all tunnels of all tunnel number 1 knots corresponds to the set of orbits of vertices of  $\mathcal{D}(H)$  under  $\mathcal{G}$ . So it is natural to examine the quotient complex  $\mathcal{D}(H)/\mathcal{G}$ , which is illustrated in figure 2. Through work of the first author [3], this action is well-understood. A *primitive* disk in  $H$  is a disk  $D$  such that there is a disk

FIGURE 2. A portion of  $\mathcal{D}(H)/\mathcal{G}$  and  $\mathcal{T}$  near the primitive orbits.

$E$  in  $\overline{S^3 - H}$  for which  $\partial D$  and  $\partial E$  intersect transversely in one point in  $\partial H$ . The primitive disks (regarded as vertices) span a contractible subcomplex  $\mathcal{P}(H)$  of  $\mathcal{D}(H)$ , called the *primitive subcomplex*. The action of  $\mathcal{G}$  on  $\mathcal{P}(H)$  is as transitive as possible, indeed the quotient  $\mathcal{P}(H)/\mathcal{G}$  is a single 2-simplex  $\Pi$  which is the image of any 2-simplex of the first barycentric subdivision of  $\mathcal{P}(H)$ . Its vertices are  $\pi_0$ , the orbit of all primitive disks,  $\mu_0$ , the orbit of all pairs of disjoint primitive disks, and  $\theta_0$ , the orbit of all triples of disjoint primitive disks.

On the remainder of  $\mathcal{D}(H)$ , the stabilizers of the action are as small as possible. A 2-simplex which has two primitive vertices and one nonprimitive is identified with some other such simplices, then folded in half and attached to  $\Pi$  along the edge  $\langle \mu_0, \pi_0 \rangle$ . The nonprimitive vertices of such 2-simplices are exactly the disks in  $\mathcal{D}(H)$  that are disjoint from some primitive pair, and these are called *simple* disks. As tunnels, they are the upper and lower tunnels of 2-bridge knots. The remaining 2-simplices of  $\mathcal{D}(H)$  receive no self-identifications, and descend to portions of  $\mathcal{D}(H)/\mathcal{G}$  that are treelike and are attached to one of the edges  $\langle \pi_0, \tau_0 \rangle$  where  $\tau_0$  is simple.

The tree  $\tilde{\mathcal{T}}$  shown in figure 1 is constructed as follows. Let  $\mathcal{D}'(H)$  be the first barycentric subdivision of  $\mathcal{D}(H)$ . Denote by  $\tilde{\mathcal{T}}$  the subcomplex of  $\mathcal{D}'(H)$  obtained by removing the open stars of the vertices of  $\mathcal{D}(H)$ . It is a bipartite graph, with “white” vertices of valence 3 represented by triples and “black” vertices of (countably) infinite valence represented by pairs. The valences reflect the fact that moving along an edge from a triple to a pair corresponds to removing one of its three disks, while moving from a pair to a triple corresponds to adding one of infinitely many possible third disks to a pair. The possible disjoint third disks that can be added are called the *slope disks* for the pair.

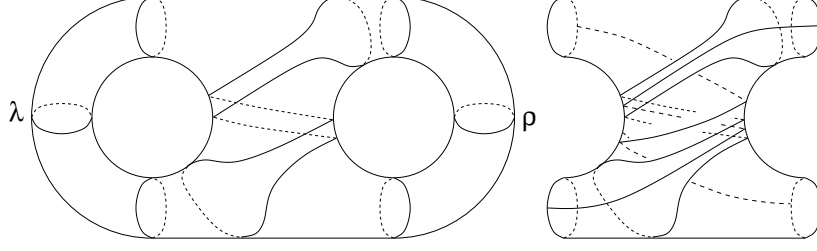


FIGURE 3. A slope disk and one of its cabling arcs.

The image  $\tilde{T}/\mathcal{G}$  of  $\tilde{T}$  in  $\mathcal{D}(H)/\mathcal{G}$  is a tree  $\mathcal{T}$ . The vertices of  $\mathcal{D}(H)/\mathcal{G}$  that are images of vertices of  $\mathcal{D}(H)$  are not in  $\mathcal{T}$ , but their links in  $\mathcal{D}'(H)/\mathcal{G}$  are subcomplexes of  $\mathcal{T}$ . These links are infinite trees. For each such vertex  $\tau$  of  $\mathcal{D}(H)/\mathcal{G}$ , i. e. each tunnel, there is a unique shortest path in  $\mathcal{T}$  from  $\theta_0$  to the vertex in the link of  $\tau$  that is closest to  $\theta_0$ . This path is called the *principal path* of  $\tau$ , and this closest vertex is a triple, called the *principal vertex* of  $\tau$ . The two disks in the principal vertex, other than  $\tau$ , are called the *principal meridian pair* of  $\tau$ . They are exactly the disks called  $\mu^+$  and  $\mu^-$  that play a key role in [11]. Figure 8 below shows the principal path of a certain tunnel.

## 2. SLOPE DISKS

In [4], it is explained how moving along the principal path of a tunnel  $\tau$  encodes a sequence of cabling constructions starting from the tunnel of the trivial knot and producing a sequence of tunnels ending with  $\tau$ . We will review the cabling construction in section 3 below. Each cabling is determined by a rational parameter, called its *slope*. In section 9 below, we will compute these slopes for many tunnels of torus knots, so it is necessary to recall the precise definition of slope. For the other applications, the precise details of the definition are not needed, so a more superficial reading of this section might suffice.

Fix a pair of disks  $\lambda$  and  $\rho$  (for “left” and “right”) in  $H$ , as shown abstractly in figure 3. Of course, in the true picture in  $H$  in  $S^3$ , these can look a great deal more complicated than the primitive pair shown in figure 3. Let  $B$  be  $H$  cut along  $\lambda \cup \rho$ . The frontier of  $B$  in  $H$  consists of four disks which appear vertical in figure 3. Denote this frontier by  $F$ , and let  $\Sigma$  be  $B \cap \partial H$ , a sphere with four holes. A *slope disk for  $\{\lambda, \rho\}$*  is an essential disk, possibly separating, which is contained in  $B$  and not isotopic to any component of  $F$ . The boundary of a slope disk always separates  $\Sigma$  into two pairs of pants, conversely any loop in  $\Sigma$  that is not homotopic into  $\partial\Sigma$  is the boundary of a unique slope disk. If two slope disks are isotopic in  $H$ , then they are isotopic in  $B$ .

An arc in  $\Sigma$  whose endpoints lie in two different boundary circles of  $\Sigma$  is called a *cabling arc*. Figure 3 shows a pair of cabling arcs disjoint from a

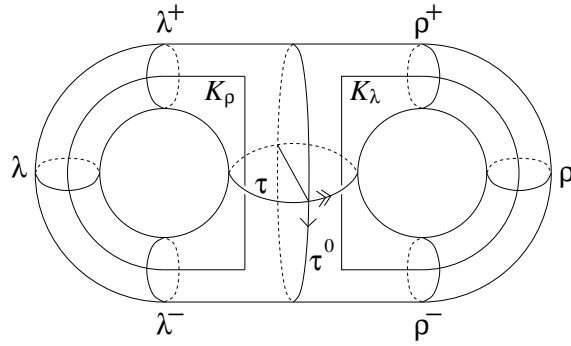


FIGURE 4. The slope-zero perpendicular disk  $\tau^0$ . It is chosen so that  $K_\lambda$  and  $K_\rho$  have linking number 0.

slope disk. A slope disk is disjoint from a unique pair of cabling arcs, and each cabling arc determines a unique slope disk.

Each choice of nonseparating slope disk for a pair  $\mu = \{\lambda, \rho\}$  determines a correspondence between  $\mathbb{Q} \cup \{\infty\}$  and the set of all slope disks of  $\mu$ , as follows. Fixing a nonseparating slope disk  $\tau$  for  $\mu$ , write  $(\mu; \tau)$  for the ordered pair consisting of  $\mu$  and  $\tau$ .

**Definition 2.1.** A *perpendicular disk* for  $(\mu; \tau)$  is a disk  $\tau^\perp$ , with the following properties:

- (1)  $\tau^\perp$  is a slope disk for  $\mu$ .
- (2)  $\tau$  and  $\tau^\perp$  intersect transversely in one arc.
- (3)  $\tau^\perp$  separates  $H$ .

There are infinitely many choices for  $\tau^\perp$ , but because  $H \subset S^3$  there is a natural way to choose a particular one, which we call  $\tau^0$ . It is illustrated in figure 4. To construct it, start with any perpendicular disk and change it by Dehn twists of  $H$  about  $\tau$  until the core circles of the complementary solid tori have linking number 0 in  $S^3$ .

For calculations, it is convenient to draw the picture as in figure 4, and orient the boundaries of  $\tau$  and  $\tau^0$  so that the orientation of  $\tau^0$  (the “ $x$ -axis”), followed by the orientation of  $\tau$  (the “ $y$ -axis”), followed by the outward normal of  $H$ , is a right-hand orientation of  $S^3$ . At the other intersection point, these give the left-hand orientation, but the coordinates are unaffected by changing the choices of which of  $\{\lambda, \rho\}$  is  $\lambda$  and which is  $\rho$ , or changing which of the disks  $\lambda^+$ ,  $\lambda^-$ ,  $\rho^+$ , and  $\rho^-$  are “+” and which are “-”, provided that the “+” disks both lie on the same side of  $\lambda \cup \rho \cup \tau$  in figure 4.

Let  $\tilde{\Sigma}$  be the covering space of  $\Sigma$  such that:

- (1)  $\tilde{\Sigma}$  is the plane with an open disk of radius  $1/8$  removed from each point with half-integer coordinates.
- (2) The components of the preimage of  $\tau$  are the vertical lines with integer  $x$ -coordinate.

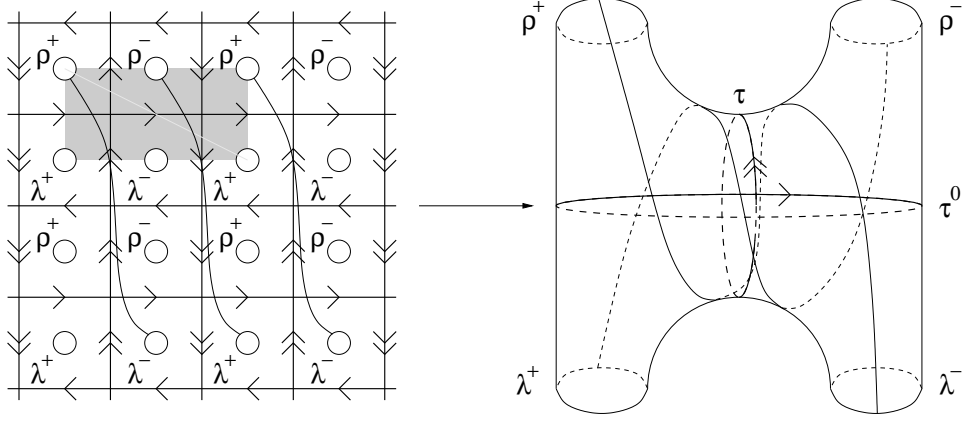


FIGURE 5. The covering space  $\tilde{\Sigma} \rightarrow \Sigma$ , and some lifts of a  $[1, -3]$ -cabling arc. The shaded region is a fundamental domain.

- (3) The components of the preimage of  $\tau^0$  are the horizontal lines with integer  $y$ -coordinate.

Figure 5 shows a picture of  $\tilde{\Sigma}$  and a fundamental domain for the action of its group of covering transformations, which is the orientation-preserving subgroup of the group generated by reflections in the half-integer lattice lines (that pass through the centers of the missing disks). Each circle of  $\partial\tilde{\Sigma}$  double covers a circle of  $\partial\Sigma$ .

If we lift any cabling arc in  $\Sigma$  to  $\tilde{\Sigma}$ , the lift runs from a boundary circle of  $\tilde{\Sigma}$  to one of its translates by a vector  $(p, q)$  of signed integers, defined up to multiplication by the scalar  $-1$ . Thus each cabling arc receives a *slope pair*  $[p, q] = \{(p, q), (-p, -q)\}$ , and is called a  $[p, q]$ -cabling arc. The corresponding slope disk receives the slope pair  $[p, q]$  as well.

An important observation is that a  $[p, q]$ -slope disk is nonseparating in  $H$  if and only if  $q$  is odd. Both happen exactly when the corresponding cabling arc has one endpoint in  $\lambda^+$  or  $\lambda^-$  and the other in  $\rho^+$  or  $\rho^-$ .

**Definition 2.2.** The  $(\mu; \tau)$ -slope of a  $[p, q]$ -slope disk or cabling arc is  $q/p \in \mathbb{Q} \cup \{\infty\}$ .

The  $(\mu; \tau)$ -slope of  $\tau^0$  is 0, and the  $(\mu; \tau)$ -slope of  $\tau$  is  $\infty$ .

Slope disks for a *primitive* pair are handled in a special way. Rather than using a particular choice of  $\tau$  from the context, one chooses  $\tau$  to be some third primitive disk. Altering this choice can change  $[p, q]$  to any  $[p + nq, q]$ , but the quotient  $p/q$  is well-defined as an element of  $\mathbb{Q}/\mathbb{Z} \cup \{\infty\}$ . This element  $[p/q]$  is called the *simple slope* of the slope disk (it is  $[0]$  exactly when the slope disk is itself primitive). Two simple disks have the same simple slope exactly when they are equivalent by an element of the Goeritz group.



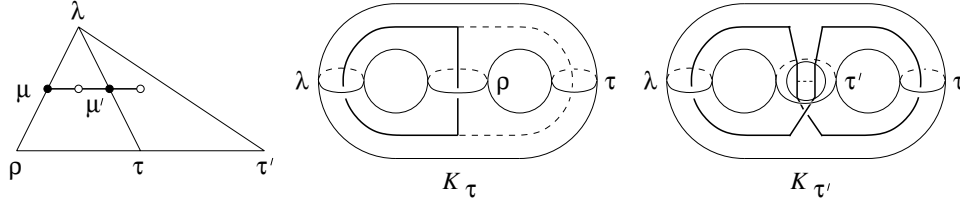


FIGURE 6. Schematic for the general cabling construction. In the middle ball in the right-hand picture of  $H$ , the two vertical arcs form some rational tangle, disjoint from the disk  $\tau'$ .

### 3. PARAMETERIZATION AND CABLING OPERATIONS

In this section, we discuss the Parameterization Theorem from [4]. First, we review the cabling construction.

In a sentence, the cabling construction is to “Think of the union of  $K$  and the tunnel arc as a  $\theta$ -curve, and rationally tangle the ends of the tunnel arc and one of the arcs of  $K$  in a neighborhood of the other arc of  $K$ .” We sometimes call this “swap and tangle,” since one of the arcs in the knot is exchanged for the tunnel arc, then the ends of other arc of the knot and the tunnel arc are connected by a rational tangle.

Figure 6 illustrates the cabling construction schematically. Begin with a pair  $\mu = \{\lambda, \rho\}$  and a triple  $\mu \cup \{\tau\}$ . In a  $\theta$ -curve corresponding to  $\mu \cup \{\tau\}$ , the union of the arcs dual to  $\mu$  is  $K_\tau$ , and the arc dual to  $\tau$  is a tunnel arc for  $K_\tau$ . Moving through  $\mathcal{T}$  starting at the edge  $\langle \mu, \mu \cup \{\tau\} \rangle$  determines a sequence of steps in which one of the two disks of a pair  $\{\lambda, \rho\}$  is replaced by a tunnel disk  $\tau$ , and a slope disk  $\tau'$  of the new pair  $\mu'$  (with  $\tau'$  nonseparating in  $H$ ) is chosen as the new tunnel disk, ending up at the edge  $\langle \mu', \mu' \cup \{\tau'\} \rangle$ . This is a cabling operation producing  $\tau'$  from  $\tau$ . It is required that  $\tau' \neq \tau$ , that is, cablings do not allow one to “backtrack” in  $\mathcal{T}$ .

As illustrated in figure 6, the way that the path determines the particular cabling operation is:

- (1) The selection of  $\lambda$  or  $\rho$  corresponds to which edge one chooses to move out of the white vertex  $\{\lambda, \rho, \tau\}$ .
- (2) The selection of the new slope disk  $\tau'$  corresponds to which edge one chooses to continue out of the black vertex  $\mu'$ .

Figure 7 shows the effects of a specific sequence of two cabling constructions, starting with the trivial knot and obtaining the trefoil, then a cabling construction starting with the tunnel of the trefoil.

When  $\mu$  is the primitive pair  $\mu_0$ , and  $\tau_0$  is a simple disk for  $\mu_0$ , the pair  $(\mu_0; \tau_0)$  determines a cabling construction starting with the tunnel of the trivial knot and producing  $\tau_0$ , which is an upper or lower tunnel of a 2-bridge knot. This is a *simple cabling of slope  $m_0$* , where  $m_0$  is the simple slope of  $\tau_0$ .

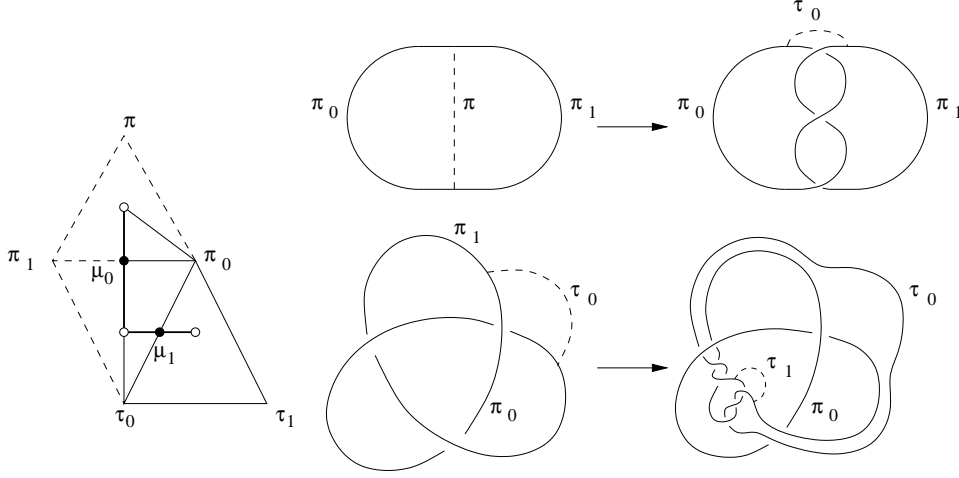


FIGURE 7. Examples of the cabling construction.

The principal path of a tunnel  $\tau$  determines a sequence of cablings, which produce  $\tau$  from the trivial tunnel  $\pi_0$ . The first is a simple cabling of some simple slope  $m_0 \in \mathbb{Q}/\mathbb{Z}$ . For  $i \geq 1$ , the cabling producing  $\tau_i$  from  $\tau_{i-1}$  obtains a rational slope  $m_i$  as follows. Let  $\sigma$  be the unique disk of  $\mu_{i-1} - \mu_i$  (the “trailing” disk), which in the schematic picture of figure 6 happens to be  $\rho$ . The slope of this cabling is defined to be the rational number  $m_i$  which is the  $(\mu_i; \sigma)$ -slope of  $\tau_i$ . Note that an illegal “backtrack” cabling would have slope  $\infty$ , since the  $(\mu; \sigma)$ -slope pair of  $\sigma$  is  $[0, 1]$ .

The Unique Cabling Sequence Theorem, theorem 13.2 of [4], states that there is a unique sequence of cabling constructions starting with the tunnel of the trivial knot and ending with  $\tau$ . It is an immediate consequence of the fact that  $\mathcal{T}$  is a tree. Viewed in terms of slope, this becomes the following result, theorem 12.3 of [4]:

**Parameterization Theorem.** *Let  $\tau$  be a knot tunnel with principal path  $\theta_0, \mu_0, \mu_0 \cup \{\tau_0\}, \mu_1, \dots, \mu_n, \mu_n \cup \{\tau_n\}$ . Fix a lift of the principal path to  $\mathcal{D}(H)$ , so that each  $\mu_i$  corresponds to an actual pair of disks in  $H$ .*

- (1) *If  $\tau$  is primitive, put  $m_0 = [0] \in \mathbb{Q}/\mathbb{Z}$ . Otherwise, let  $m_0 = [p_0/q_0] \in \mathbb{Q}/\mathbb{Z}$  be the simple slope of  $\tau_0$ .*
- (2) *If  $n \geq 1$ , then for  $1 \leq i \leq n$  let  $\sigma_i$  be the unique disk in  $\mu_{i-1} - \mu_i$  and let  $m_i = q_i/p_i \in \mathbb{Q}$  be the  $(\mu_i; \sigma_i)$ -slope of  $\tau_i$ .*
- (3) *If  $n \geq 2$ , then for  $2 \leq i \leq n$  define  $s_i = 0$  or  $s_i = 1$  according to whether or not the unique disk of  $\mu_i \cap \mu_{i-1}$  equals the unique disk of  $\mu_{i-1} \cap \mu_{i-2}$ .*

*Then, sending  $\tau$  to the pair  $((m_0, \dots, m_n), (s_2, \dots, s_n))$  is a bijection from the set of all tunnels of all tunnel number 1 knots to the set of all elements*

$$((p_0/q_0], q_1/p_1, \dots, q_n/p_n), (s_2, \dots, s_n)) \text{ in} \\ (\mathbb{Q}/\mathbb{Z}) \cup (\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}) \cup (\cup_{n \geq 2} \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}^n \times C_2^{m-1})$$

with all  $q_i$  odd.

**Remark 3.1.** Up to details of definition, the final slope  $m_n$  in the Parameterization Theorem is the Scharlemann-Thompson invariant [11].

A tunnel produced from the tunnel of the trivial knot by a single cabling construction is called a *simple* tunnel. As already noted, these are the “upper and lower” tunnels of 2-bridge knots. According to the Parameterization Theorem, these are determined by a single  $\mathbb{Q}/\mathbb{Z}$ -valued parameter  $m_0$ , and this is of course a version of the standard rational parameter that classifies the 2-bridge knot.

A tunnel is called *semisimple* if it is disjoint from a primitive disk, but not from any primitive pair. The simple and semisimple tunnels are exactly the  $(1, 1)$ -tunnels, that is, the tunnels that can be put into 1-bridge position with respect to a Heegard torus of  $S^3$ . The non-simple tunnels of 2-bridge knots are semisimple, and in [4], their parameter sequences are calculated. Since they are semisimple, their parameters  $s_2, \dots, s_n$  in the Parameterization Theorem are all 0. Their slope parameters are determined by a somewhat complicated, but easily programmable algorithm using the continued fraction expansion of the classifying parameter.

Finally, a tunnel is called *regular* if it is neither primitive, simple, or semisimple.

#### 4. DISTANCE AND DEPTH

In this section we formally introduce the distance and depth invariants of a tunnel  $\tau$ . The (*Hempel*) *distance*  $\text{dist}(\tau)$  is the shortest distance in the curve complex of  $\partial H$  from  $\partial\tau$  to a loop that bounds a disk in  $\overline{S^3 - H}$  (see J. Johnson [7] and Y. Minsky, Y. Moriah, and S. Schleimer [8]). It is well-defined since the action of the Goeritz group on  $\partial H$  preserves the set of loops that bound disks in  $H$  and the set that bound in  $\overline{S^3 - H}$ .

A nonseparating disk has distance 1 if and only if it is primitive, since both conditions are equivalent to the condition that cutting  $H$  along the disk produces an unknotted solid torus (see [7, Section 4]). Therefore the tunnel of the trivial knot is the only tunnel of distance 1. A simple or semisimple tunnel has distance 2, since it is disjoint from a primitive disk. There are, however, regular tunnels of distance 2. In section 9 we will see that the regular tunnels of torus knots all have distance 2.

Recall that if  $\Sigma = (\overline{H - \text{Nbd}(K_\tau)}, \overline{S^3 - H})$  is a Heegaard splitting of the complement of  $K_\tau$ , then the (*Hempel*) *distance*  $\text{dist}(\Sigma)$  is the minimal distance in the curve complex of  $\partial H$  between the boundary of a disk in  $\overline{H - \text{Nbd}(K_\tau)}$  and the boundary of a disk in  $\overline{S^3 - H}$  (where the disks may be separating). Clearly,  $\text{dist}(\Sigma) \leq \text{dist}(\tau)$ . On the other hand, Johnson [7, Lemma 11] proved that

**Lemma 4.1** (Johnson).  $\text{dist}(\tau) \leq \text{dist}(\Sigma) + 1$ .

M. Scharlemann and M. Tomova [12] proved the following stability result:

**Theorem 4.2** (Scharlemann-Tomova). *Genus- $g$  Heegaard splittings of distance more than  $2g$  are isotopic.*

Using lemma 4.1 and theorem 4.2, Johnson [7, Corollary 13] deduced the following:

**Theorem 4.3** (Johnson). *If  $\tau$  is a tunnel of a tunnel number 1 knot  $K_\tau$  and  $\text{dist}(\tau) > 5$ , then  $\tau$  is the unique tunnel of  $K_\tau$ .*

Theorem 15.2 of [4], an immediate consequence of the Parameterization Theorem, determines all orientation-reversing self-equivalences of tunnels:

**Theorem 4.4.** *Let  $\tau$  be a tunnel of a tunnel number 1 knot or link. Suppose that  $\tau$  is equivalent to itself by an orientation-reversing equivalence. Then  $\tau$  is the tunnel of the trivial knot, the trivial link, or the Hopf link.*

Combining theorems 4.3 and 4.4 gives the following:

**Corollary 4.5.** *If  $\tau$  is a tunnel of a tunnel number 1 knot  $K_\tau$  and  $\text{dist}(\tau) > 5$ , then  $K_\tau$  is not amphichiral.*

For theorem 4.4 shows that an orientation-reversing equivalence from  $K_\tau$  to  $K_\tau$  would produce a second tunnel for  $K_\tau$ .

Distance also has implications for hyperbolicity.

**Theorem 4.6.** *If  $K_\tau$  is a torus knot or a satellite knot, then  $\text{dist}(\tau) \leq 2$ . Consequently, if  $\text{dist}(\tau) \geq 3$ , then  $K_\tau$  is hyperbolic.*

*Proof.* We have already mentioned the fact, verified in section 9 below, that the regular tunnels of torus knots have distance 2. The other tunnels of torus knots are simple or semisimple, so also have distance 2. K. Morimoto and M. Sakuma [9] found all tunnels of tunnel number 1 satellite knots, showing in particular that they are semisimple.  $\square$

The *depth* of  $\tau$  is the simplicial distance  $\text{depth}(\tau)$  in the 1-skeleton of  $\mathcal{D}(H)/\mathcal{G}$  from  $\tau$  to the primitive vertex  $\pi_0$ . The inequality

$$\text{dist}(\tau) - 1 \leq \text{depth}(\tau)$$

mentioned in the introduction is immediate from the definitions. Also from the definitions,  $\tau$  is primitive if and only if  $\text{depth}(\tau) = 0$ , is simple or semisimple if and only if  $\text{depth}(\tau) = 1$ , and is regular if and only if  $\text{depth}(\tau) \geq 2$ . As already mentioned, in section 9 we will see a sequence of torus knot tunnels of distance 2 with depths that grow arbitrarily large.

## 5. GIANT STEPS

**Definition 5.1.** Let  $\tau$  and  $\tau'$  be tunnels of a tunnel number 1 knot. We say that  $\tau'$  is obtained from  $\tau$  by a *giant step* if  $\tau$  and  $\tau'$  are the endpoints of a 1-simplex of  $\mathcal{D}(H)/\mathcal{G}$ . Equivalently,  $\tau$  and  $\tau'$  can be represented by disjoint disks in  $H$ .

It is clear that the depth of a tunnel is the minimum number of giant steps needed to transform the tunnel to the tunnel of the trivial knot, or vice versa.

In [6], Goda, Scharlemann, and Thompson gave a geometric definition of giant steps (this is one reason for our selection of the name Giant Steps), as follows. Let  $\tau$  be a nonseparating disk in  $H$ , and let  $K$  be a simple closed curve in  $\partial H$  that intersects  $\tau$  transversely in one point. Let  $N$  be a regular neighborhood in  $H$  of  $K \cup \tau$ . Then the frontier of  $N$  separates  $H$  into two solid tori, one a regular neighborhood of  $K$ , so  $K$  is a tunnel number 1 knot.

In the previous construction, the meridian disk  $\tau'$  of the solid torus that does not contain  $K \cup \tau$  is the *unique* nonseparating disk  $\tau'$  in  $H$  that is disjoint from  $K \cup \tau$ , and  $\tau'$  is a tunnel of  $K$ . That is, the construction produces a specific tunnel of the resulting knot  $K$ . A giant step as we have defined it simply amounts to choosing the  $\tau'$  first;  $K$  is then determined up to isotopy in  $H$  and in  $S^3$ , although not up to isotopy in  $\partial H$ .

Since the complex  $\mathcal{D}(H)/\mathcal{G}$  is connected, we have the following, which is part of Proposition 1.11 of [6].

**Proposition 5.2.** *Let  $\tau$  be a tunnel of a tunnel number 1 knot. Then there is a sequence of giant steps that starts with the tunnel of the trivial knot and ends with  $\tau$ .*

## 6. MINIMAL SEQUENCES OF GIANT STEPS

In this section, we will calculate the number of minimal length sequences of giant steps that start from  $\pi_0$ , the tunnel of the trivial knot, and end with a given tunnel  $\tau$ . First, we will observe that any such minimal sequence corresponds to a minimal length simplicial path in the 1-skeleton of a neighborhood of the principal path of  $\tau$ . An elementary counting method then gives an algorithm to calculate the number of such paths.

By a *path* (between two vertices) in  $\mathcal{D}(H)/\mathcal{G}$ , we mean a simplicial path in the 1-skeleton of  $\mathcal{D}(H)/\mathcal{G}$ , passing through a sequence of vertices that are images of vertices of  $\mathcal{D}(H)$  (i. e. vertices that represent tunnels). We describe such a path simply by listing the vertices through which it passes. From section 5, we know that the minimal sequences of giant steps from the trivial tunnel to a given tunnel correspond exactly to the minimal-length paths in  $\mathcal{D}(H)/\mathcal{G}$  from  $\pi_0$  to the tunnel vertex. We will only be interested in minimal-length paths.

**Definition 6.1.** Let  $\tau$  be a nontrivial tunnel. Define the *corridor* of  $\tau$ ,  $C(\tau)$ , as follows. Write the vertices of the principal path of  $\tau$  as  $\theta_0, \mu_0$ ,

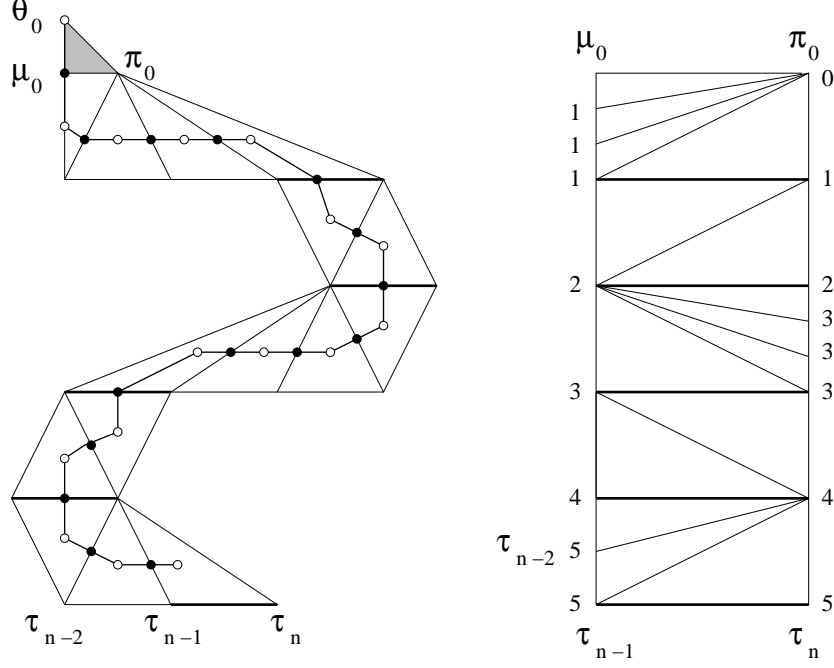


FIGURE 8. The principal path and corridor  $C(\tau_n)$  of a tunnel  $\tau_n$ . The  $\nabla$ -edges are emphasized. In the picture of  $C(\tau_n)$  on the right, the depths of the tunnels are labeled.

$\mu_0 \cup \tau_0, \mu_1, \mu_1 \cup \tau_1, \dots, \mu_n \cup \tau_n$ , where  $\tau = \tau_n$ . Then  $C(\tau)$  is the union of the 2-simplices whose barycenters are the  $\mu_i \cup \tau_i$  for  $0 \leq i \leq n$  (where  $\mu_0 \cup \tau_0$  is regarded as the barycenter of the 2-simplex spanned by  $\pi_0, \mu_0$ , and  $\tau_0$ ).

When  $\tau$  is a simple tunnel,  $C(\tau)$  is the triangle  $\langle \pi_0, \mu_0, \tau_0 \rangle$ . Otherwise, it can be viewed as a rectangular or trapezoidal strip with end edges  $\langle \pi_0, \mu_0 \rangle$  and  $\langle \tau_j, \tau_n \rangle$  for some  $j$ , as in the figure on the right in figure 8.

**Lemma 6.2.** *Let  $\tau$  be a tunnel, and let  $\sigma_0, \sigma_1, \dots, \sigma_n$  be a path in  $\mathcal{D}(H)/\mathcal{G}$  of minimal length among the paths connecting  $\sigma_0$  to  $\sigma_n$ . If  $\sigma_0$  and  $\sigma_n$  lie in  $C(\tau)$ , then each  $\sigma_i$  lies in  $C(\tau)$ .*

*Proof.* If the lemma is false, then there exist  $i$  and  $j$  with  $0 < i < i+1 < j < n$  for which  $\sigma_i$  and  $\sigma_j$  lie in the frontier in  $\mathcal{D}(H)/\mathcal{G}$  of  $C(\tau)$ , but  $\sigma_k$  does not lie in  $C(\tau)$  for any  $m$  with  $i < k < j$ . Let  $\sigma'_i$  and  $\sigma''_i$  be the vertices of  $\mathcal{D}(H)/\mathcal{G}$  adjacent to  $\sigma_i$  on the frontier of  $C(\tau)$ . The union of 1-simplices  $\langle \sigma'_i, \sigma_i \rangle$  and  $\langle \sigma_i, \sigma''_i \rangle$  separates  $\mathcal{D}(H)/\mathcal{G}$ , indeed every 1-simplex of  $\mathcal{D}(H)/\mathcal{G}$  separates. Therefore  $\sigma_j$  must equal one of  $\sigma_i, \sigma'_i$ , or  $\sigma''_i$ . But this implies that the original path did not have minimal length.  $\square$

In the special case that  $\tau$  is of depth 1,  $\tau$  lies in the link in  $\mathcal{D}(H)/\mathcal{G}$  of  $\pi_0$ , and there is clearly a unique path of length 1 from  $\tau$  to  $\pi_0$ . From now on, we assume that  $\tau$  has depth at least 2.

Now, regard  $C(\tau)$  as in the diagram on the right in figure 8, with the edge  $\langle \mu_0, \pi_0 \rangle$  on top, and with  $\tau$  as one of the endpoints of the bottom edge. In the triangulation of  $C(\tau)$ , a  $\nabla$ -edge of depth  $n$  is an edge whose endpoints have depth  $n$  and lie on different sides of  $C(\tau)$ , and for which all vertices lying below its endpoints on either side have depth greater than  $n$ . In figure 8, the  $\nabla$ -edges are highlighted.

Since the endpoints of any edge of  $C(\tau)$  can have depths that differ by at most 1, there exists a unique  $\nabla$ -edge  $\nabla(i)$  in  $C(\tau)$  of depth  $i$  for each  $i$  with  $1 \leq i < \text{depth}(\tau)$  (and there may be one of depth  $n$ ).

The name  $\nabla$ -edges arises from the fact that these edges are the tops of 2-simplices of the corridor that appear as  $\nabla$ 's when the corridor is drawn with depth corresponding to the vertical coordinate, as in the diagram on the left in figure 8. Every nonprimitive 2-simplex of  $\mathcal{D}(H)/\mathcal{G}$  has two vertices of the same depth and a third of depth either 1 larger or 1 smaller than that depth; for a “ $\nabla$ ” 2-simplex that depth is 1 larger, while it is 1 smaller for a “ $\Delta$ ” 2-simplex.

Denote the left and right endpoints of  $\nabla(i)$  by  $\partial_L(\nabla(i))$  and  $\partial_R(\nabla(i))$  respectively.

**Lemma 6.3.** *Let  $\nabla(i-1)$  and  $\nabla(i)$  be successive  $\nabla$ -edges. Then at least one of the pairs  $\{\partial_L(\nabla(i-1)), \partial_L(\nabla(i))\}$  and  $\{\partial_R(\nabla(i-1)), \partial_R(\nabla(i))\}$  are the endpoints of an edge that lies in a side of  $C(\tau)$ .*

*Proof.* For each endpoint of  $\nabla(i)$ , select a path of length  $i$  from the endpoint to  $\pi_0$ . By lemma 6.2, these paths lie in  $C(\tau)$ . In particular, each of their first edges connects an endpoint of  $\nabla(i)$  to an endpoint of  $\nabla(i-1)$ . At most one of these first edges can be diagonal, so at least one lies in a side.  $\square$

Lemma 6.3 shows that the triangulation of the portion of  $C$  between  $e$  and  $f$  must have one of the four configurations  $L_1$ ,  $R_1$ ,  $L_2$ , or  $R_2$  shown in figure 9. The portion of  $C(\tau)$  above  $\nabla(1)$  must be as in the leftmost diagram in figure 9, where there may be only one 2-simplex above the diagonal.

Now, we show how to calculate the number of minimal paths from  $\tau$  to  $\pi_0$ . Denote by  $\lambda_i$  the number of paths in  $C(\tau)$  of length  $\text{depth}(\tau) - i$  from  $\tau$  to the left endpoint of  $\nabla(i)$ , and by  $\rho_i$  the number to its right endpoint. From figure 9, we see that the total number of paths from  $\tau$  to  $\pi_0$  is  $\lambda_1 + \rho_1$ .

Let  $\nabla(n)$  be the  $\nabla$ -edge in  $C(\tau)$  of maximum depth. If  $\tau$  is the left (or right) endpoint of  $\nabla(n)$ , then  $\begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix}$  is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (or  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ). If  $\tau$  and the endpoints of  $\nabla(n)$  span a 2-simplex, as in the case of the tunnel  $\tau_{n-2}$  in figure 8, then  $\begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix}$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Otherwise, there is exactly one edge from  $\sigma$  to an endpoint of  $\nabla(n)$ , and  $\begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix}$  is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  according to the endpoint.

For each  $2 \leq i \leq n$ , let  $C_i$  be  $L_1$ ,  $R_1$ ,  $L_2$ , or  $R_2$  according to which of the four configurations in figure 9 describes the triangulation of  $C$  between

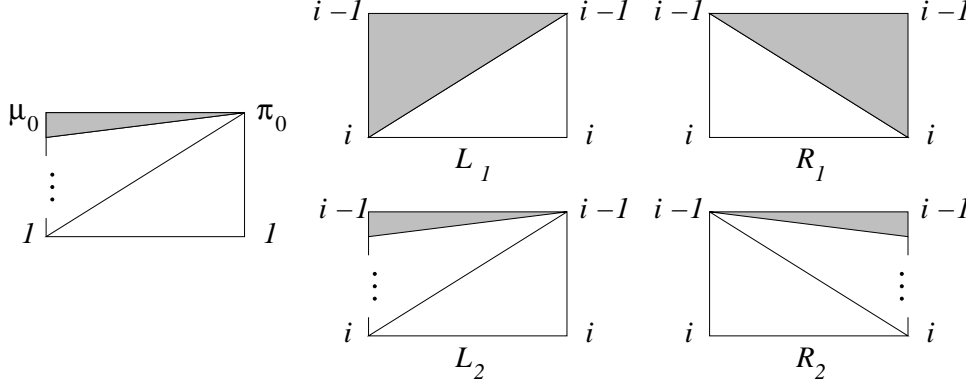


FIGURE 9. The configuration above  $\nabla(1)$ , and the four possible configurations between two  $\nabla$ -edges. In  $L_1$  and  $R_1$  there is only one 2-simplex above the diagonal edge, while in  $L_2$  and  $R_2$  there are two or more. In the configuration above  $\nabla(1)$ , there may be only one 2-simplex above the diagonal. The shaded 2-simplices are  $\nabla$  2-simplices. The letter  $L$  (respectively,  $R$ ) signifies that the  $\nabla$  2-simplex has an edge on the left side (respectively, right side) of the corridor.

$\nabla(i-1)$  and  $\nabla(i)$ . For  $2 \leq i \leq n-1$ , put  $M_i$  equal to the matrix given in the following table, according to the value of  $C_i$ :

$C_i$	$L_1$	$R_1$	$L_2$	$R_2$
$M_i$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Observe that

$$M_i \begin{pmatrix} \lambda_i \\ \rho_i \end{pmatrix} = \begin{pmatrix} \lambda_{i-1} \\ \rho_{i-1} \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} \lambda_1 \\ \rho_1 \end{pmatrix} = M_2 M_3 \cdots M_n \begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix},$$

or alternatively, putting  $M_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ , the number of minimal paths from  $\tau$  to  $\pi_0$  is the entry of the  $1 \times 1$ -matrix  $M_1 M_2 \cdots M_n \begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix}$ .

The algorithm just described is easily implemented computationally [5]. The input is the binary string  $s_2 s_3 \cdots s_n$  of parameters from the Parameterization Theorem, which completely determine the structure of  $C(\tau)$ . The input is broken into blocks having one of the four forms 10, 11, 100+, and 110+, where 0+ indicates a nonempty string of 0's. A 10-block, for example, produces a configuration  $A_1$  when the principal path is moving in from right to left, and a configuration  $B_1$  when it is moving from left to right, and so on. The blocks 10 and 100+ reverse the direction, and the others do not. A



leftover 1 at the end of the input string indicates that the bottom triangle in  $C(\tau)$  is a  $\nabla$  2-simplex, so that  $\begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and in the other cases  $\begin{pmatrix} \lambda_n \\ \rho_n \end{pmatrix}$  is worked out from the final block (and the direction of travel of the principal path at that point).

For the example in figure 8, the input string is 0011100011100 and the output of the program is:

```
Depth> gst( '0011100011100', verbose=True )
The intermediate configurations are L1, R2, R1.
The transformation matrices are:
  [ [ 1, 0 ], [ 1, 1 ] ]
  [ [ 1, 1 ], [ 0, 0 ] ]
  [ [ 1, 1 ], [ 0, 1 ] ]
and their product is [ [ 1, 2 ], [ 1, 2 ] ].
The final block has configuration L2.
This tunnel has 4 minimal giant step constructions.
```

Among the interesting examples are the tunnels whose parameter sequences are the following:

- (1)  $s_2 s_3 \cdots s_n = 100100 \cdots 100$ . The configuration sequence alternates as  $R2, L2, R2, L2 \dots$ , and there is a unique minimal giant step sequence.
- (2)  $s_2 s_3 \cdots s_{2n+1} = 1010 \cdots 10$ . The configuration sequence alternates as  $R1, L1, R1, L1 \dots$ , and the number of minimal giant step sequences is  $a_n$ , the term in the Fibonacci sequence  $(a_0, a_1, a_2, \dots) = (1, 1, 2, 3, 5, \dots)$ .
- (3)  $s_2 s_3 \cdots s_{2n+1} = 111 \cdots 1$ , an even number of 1's. The configuration sequence is  $L1, L1 \dots$ , and there is a unique minimal giant step sequence.
- (4)  $s_2 s_3 \cdots s_{2n} = 111 \cdots 1$ , an odd number of 1's. The configuration sequence is  $L1, L1 \dots$ , but there is only a single  $\nabla$  2-simplex below  $\nabla(n)$ , and there are  $n + 1$  minimal giant step sequences.

Note that examples of the last two types are obtained from each other by a single additional cabling construction, even though the numbers of minimal giant step constructions differ by arbitrarily large amounts.

**Remark 6.4.** The algorithm shows that tunnels with a unique minimal giant step sequence are sparse. For instance, the product  $R_1 L_1 R_1 L_1$  is a matrix with all entries greater than 1, so whenever this appears as any block of four terms in the product  $M_1 M_2 \cdots M_n$  that occurs in the algorithm, there must be more than one minimal giant step sequence. Products  $M_1 M_2 \cdots M_n$  in which this block occurs are generic in any reasonable sense.

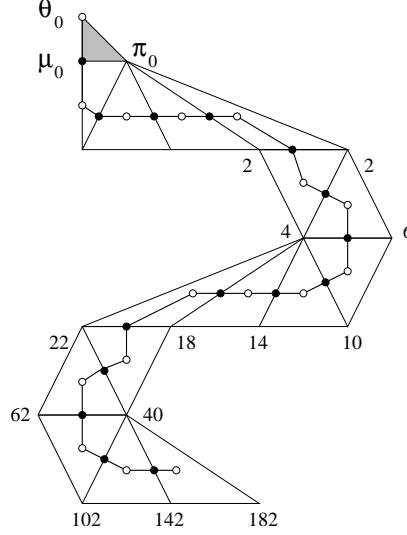


FIGURE 10.

## 7. DEPTH AND BRIDGE NUMBER

Some deep geometric results of Goda, Scharlemann, and Thompson allow us to obtain information about the bridge numbers of the knots  $K_\tau$ . They show that once one leaves the semisimple region, the bridge number grows at least exponentially with the depth (in fact, it grows rapidly with the number of cablings needed to produce the tunnel).

Our most precise result on bridge numbers is theorem 7.1, which gives a lower bound for the bridge number of a tunnel number 1 knot in terms of the principal path of any of its regular tunnels. Its statement is a bit uninviting, but it says something very simple. Figure 10 illustrates how theorem 7.1 bounds the bridge numbers of the tunnels along the principal path of the example of figure 8, assuming that the last two tunnels at depth 1 were tunnels of 2-bridge knots.

**Theorem 7.1.** *Let  $\tau_n$  be a regular tunnel with principal path  $\theta_0, \mu_0, \mu_0 \cup \{\tau_0\}, \mu_1, \dots, \mu_n, \mu_n \cup \{\tau_n\}, \tau_n$ . In the principal path of  $\tau$ , let  $\tau_m$  be the first tunnel of depth 2, with principal vertex  $\{\tau_{m-2}, \tau_{m-1}, \tau_m\}$ . Put  $c_{m-2} = \text{br}(K_{\tau_{m-2}})$  and  $c_{m-1} = \text{br}(K_{\tau_{m-1}})$ . To the vertices  $\tau_m, \tau_{m+1}, \dots, \tau_n$ , assign values  $c_k$  inductively by the rule  $c_k = c_i + c_j$ , where the principal vertex of  $\tau_k$  is  $\{\tau_i, \tau_j, \tau_k\}$ . Then the bridge number of  $K_{\tau_n}$  is at least  $c_n$ .*

Before proving theorem 7.1, we isolate the step that uses the results of Goda, Scharlemann, and Thompson [6] and Scharlemann and Thompson [11]. For its statement and proof, we remind the reader that the term *principal meridian pair* was defined near the end of section 1.

**Lemma 7.2.** *Let  $\{\lambda, \rho\}$  be the principal meridian pair of a tunnel  $\tau$ , and let  $\theta$  be the  $\theta$ -curve associated to the principal vertex  $\{\lambda, \rho, \tau\}$  of  $\tau$ . Write  $T$*

for the arc dual to  $\tau$ , and  $L$  and  $R$  for the other two arcs of  $\theta$  that are dual to  $\lambda$  and  $\rho$ , so that  $K_\tau = L \cup R$ ,  $K_\lambda = R \cup T$ , and  $K_\rho = L \cup T$ . Then there is a minimal bridge position of  $K_\tau$  in either:

- (i)  $T$  is slid to an arc in a level sphere, and  $T$  connects two bridges of  $K_\tau$ . In the  $n$ -strand trivial tangle above the level sphere, the arcs are parallel to a collection of disjoint arcs in the level sphere, which meet  $T$  only in its endpoints. Moreover,  $K_\tau \cup T$  is isotopic to the original  $\theta$ . Or,
- (ii)  $T$  is slid to an eyeglass in a level sphere. The endpoints of  $T$  can be slid slightly apart, moving  $T$  out of the level sphere, producing  $K_\tau \cup T$  isotopic to the original  $\theta$ , and showing that one of  $K_\lambda$  or  $K_\rho$  is a trivial knot, and consequently  $\tau$  is simple or semisimple.

*Proof.* By Theorem 1.8 of [6], we may move  $K_\tau \cup T$ , possibly using slide moves of  $T$  as well as isotopy, so that  $K_\tau$  is in minimal bridge position and  $T$  either lies on a level sphere and connects two bridges of  $K_\tau$ , or  $T$  is slid to an “eyeglass”. Since the leveling process involves sliding the tunnel arc  $T$ , there is *a priori* no reason for the resulting  $\theta$ -curve to be isotopic to the original  $\theta$ . But Corollary 3.4 and Theorem 3.5 (combined with Lemma 2.9) of [11] show that in (i) and (ii), the dual disks to the other two arcs of the  $\theta$ -curve are the principal meridian pair of  $\tau$ , that is,  $\lambda$  and  $\rho$ , so the resulting  $\theta$ -curve is still  $\theta$ . Finally, the description of the trivial tangle above the level sphere in case (i) is from Theorem 6.1 of [6].  $\square$

Theorem 7.1 follows immediately by application of the next lemma, which will also be used in the proof of corollary 7.4.

**Lemma 7.3.** *Let  $\tau$  be a tunnel of a nontrivial knot, and let  $\{\lambda, \rho\}$  be the principal meridian pair of  $\tau$ . Then  $\text{br}(K_\tau) \geq \text{br}(K_\lambda) + \text{br}(K_\rho) - 1$ . If  $\tau$  is regular, then  $\text{br}(K_\tau) \geq \text{br}(K_\lambda) + \text{br}(K_\rho)$ .*

*Proof.* Level a tunnel arc as in lemma 7.2. If the tunnel arc is slid to an eyeglass, pulling the endpoints slightly apart produces  $\theta$  and shows that one of  $K_\lambda$  or  $K_\rho$  is trivial. The other is in bridge position with the same number of bridges as  $K_\tau$ , so  $\text{br}(K_\tau) \geq \text{br}(K_\lambda) + \text{br}(K_\rho) - 1$ . If  $\tau$  is regular, then the eyeglass configuration cannot occur, and we see that  $\text{br}(K_\tau) \geq \text{br}(K_\lambda) + \text{br}(K_\rho)$ .  $\square$

It is straightforward to implement the iteration of theorem 7.1 computationally [5]. The only information needed for input is the sequence of parameters  $s_2, \dots, s_n$  of the Parameterization Theorem and the values  $c_{m-2}$  and  $c_{m-1}$ , called  $c_2$  and  $c_3$  in the sample output shown here:

```
Depth> bridge( '0011100011100', c2=2, c3=2, verbose=True )
The minimum bridge number of K-tau is 182.
The iteration sequence is:
2, 2, 4, 6, 10, 14, 18, 22, 40, 62, 102, 142, 182
```

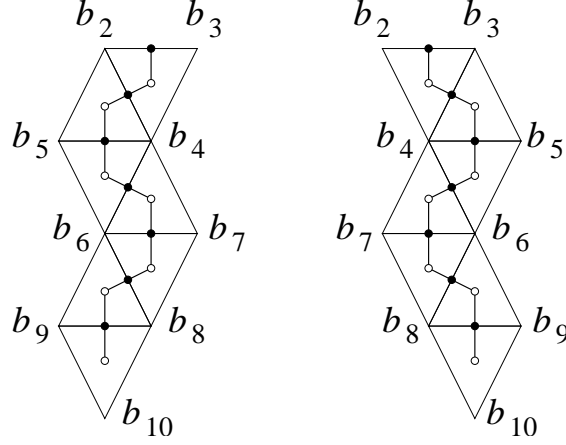


FIGURE 11. The path on the left is of cheapest descent. The path on the right is of cheapest descent if  $b_2 = b_3$ .

Theorem 7.1 gives us a general lower bound for bridge number as a function of depth:

**Corollary 7.4.** *Let  $\tau$  be a regular tunnel of depth  $d$ , and in the principal path of  $\tau$ , let  $\tau_m$  be the first tunnel of depth 2, with principal vertex  $\{\tau_{m-2}, \tau_{m-1}, \tau_m\}$ . Put  $b_2 = \text{br}(K_{\tau_{m-2}})$  and  $b_3 = \text{br}(K_{\tau_{m-1}})$ . For  $n \geq 2$  let  $b_j$  be given by the recursion*

$$\begin{aligned} b_{2n} &= b_{2n-1} + b_{2n-2} \\ b_{2n+1} &= b_{2n} + b_{2n-2} \end{aligned}$$

Then  $\text{br}(K_\tau) \geq b_{2d}$ .

*Proof.* By an application of lemma 7.3, we have  $b_2 \leq b_3$ . The left diagram in figure 11 shows a “path of cheapest descent” starting from the vertex  $\{\tau_{m-2}, \tau_{m-1}\}$  (the path in the right diagram is also a path of cheapest descent, provided that  $b_2 = b_3$ ). Any principal path having more than two tunnels at a given depth will produce an even larger bridge number, as will any principal path that emerges in the more costly direction out of a  $\nabla$  2-simplex. Applying theorem 7.1 to a path of cheapest descent gives the recursion of corollary 7.4, hence a lower bound for  $\text{br}(K_\tau)$ .  $\square$

We can now prove one of our main results.

**Minimum Bridge Number Theorem 7.5.** *For  $d \geq 1$ , the minimum bridge number of a knot having a tunnel of depth  $d$  is  $a_d$ , where  $a_1 = 2$ ,  $a_2 = 4$ , and  $a_d = 2a_{d-1} + a_{d-2}$  for  $d \geq 3$ .*

*Proof.* Taking  $b_2 = b_3 = 2$  in corollary 7.4 gives a  $b_{2d}$  which is a general lower bound for the bridge number of a tunnel at depth  $d$ , and a little bit

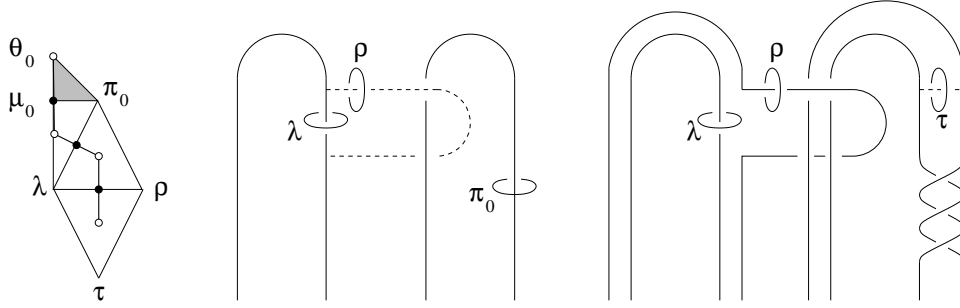


FIGURE 12.

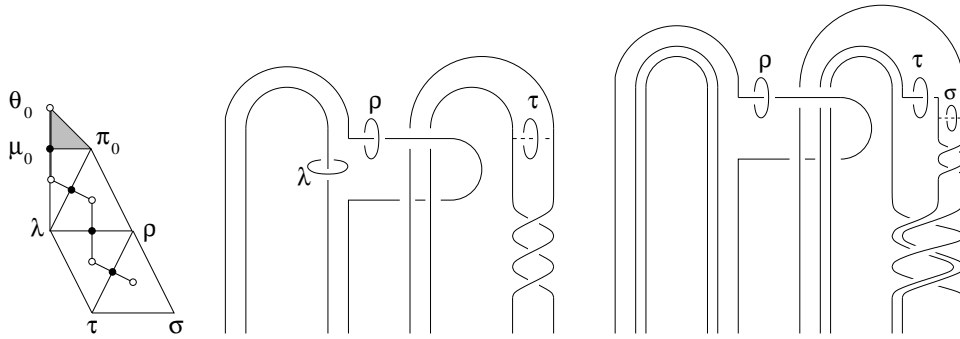


FIGURE 13.

of algebra shows that  $b_{2d} = a_d$  for the recursion in theorem 7.5. It remains to show the existence of a  $\tau$  of depth  $d$  for which  $\text{br}(K_\tau) = a_{2d}$ .

We begin with a tunnel  $\rho$  which is a semisimple tunnel of a 2-bridge knot obtained from the trivial tunnel by two cablings. Details of the construction of such a tunnel are given in section 17 of [4]. The principal path of  $\rho$  is a portion of the path shown in the leftmost diagram of figure 12, and the principal pair of  $\rho$  is  $\{\pi_0, \lambda\}$ . A tunnel  $\tau$  is constructed using a cabling as indicated in figure 12, producing a 4-bridge knot  $K_\tau$ . The number of twists in the two strands on the right is variable, it must simply be chosen so that  $K_\tau$  is a knot rather than a link. The slope of this cabling is an odd integer, since  $K_\tau$  meets the replaced disk  $\pi_0$  in only two points.

Figure 13 shows how to continue the construction. The next tunnel  $\sigma$  must be at depth 2, and at this stage (since  $b_2 = b_3$ ) we can retain either  $\lambda$  or  $\rho$ ; we have chosen to retain  $\rho$  in the example of figure 13. Again, the cabling has odd integer slope. The resulting tunnel  $\sigma$  has  $\text{br}(K_\sigma) = 6$ . Later repetitions of the construction resemble that of figure 13, but the analogues of  $K_\lambda$  and  $K_\rho$  will not have the same bridge number, and one must retain whichever disk has corresponding knot of smaller bridge number. A pattern as in figure 11 will be produced.  $\square$

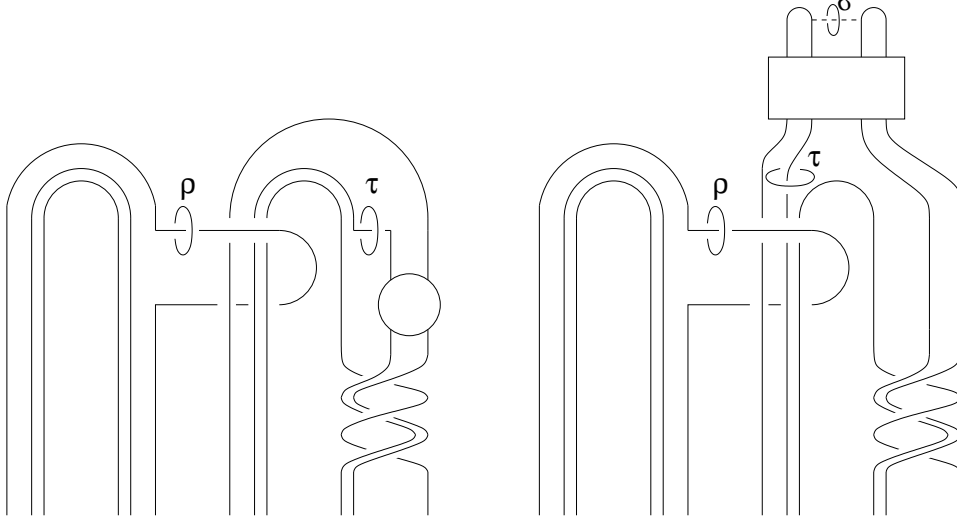


FIGURE 14.

## 8. UPPER BOUNDS FOR BRIDGE NUMBER

The construction used to prove the Minimum Bridge Number Theorem adapts to show that in general, cabling operations can be carried out efficiently from the viewpoint of bridge number. The basic idea is shown in figure 14. Its left diagram is like the right diagram of figure 13, except that a cabling of some arbitrary slope has been performed on  $K_\tau$  to produce  $K_\sigma$ ; the rational tangle in  $K_\sigma$  created by the cabling is inside a ball represented by the circle in the first diagram. We can reposition  $K_\sigma$  as in the second diagram of figure 14, by “moving the ball up to engulf infinity,” in such a way that the rectangle in the second diagram contains a 4-strand braid. The new tunnel  $\sigma$  is still level so the construction can be repeated.

To understand the effect of this construction on bridge numbers, we introduce some special terminology.

**Definition 8.1.** Let  $\{\lambda, \rho, \tau\}$  be the principal vertex of a tunnel  $\tau$ . Its dual  $\theta$ -curve has the form  $K_\tau \cup \alpha$ , where  $\alpha$  is a tunnel arc representing the tunnel  $\tau$ , and contains  $K_\lambda$  and  $K_\rho$  as subsets. Assume that  $K_\tau \cup \alpha$  is positioned so that  $K_\tau$  is in (not necessarily minimal) bridge position, and  $\alpha$  is contained in a level sphere  $S$  as in Case (1) of Figure 12 of [6]; that is, in the trivial  $n$ -strand tangle in the ball in  $S^3$  lying above  $S$ ,  $\alpha$  connects the endpoints of two different strands, and the  $n$  strands are parallel to a collection of  $n$  disjoint arcs in  $S$  that are disjoint from the interior of  $\alpha$ . We call this a *level arc position* of  $\tau$ , keeping in mind that  $K_\tau$  is not assumed to be in minimal bridge position. Note, however, that according to lemma 7.2, results of Goda, Scharlemann, and Thompson show that for a regular tunnel, we may

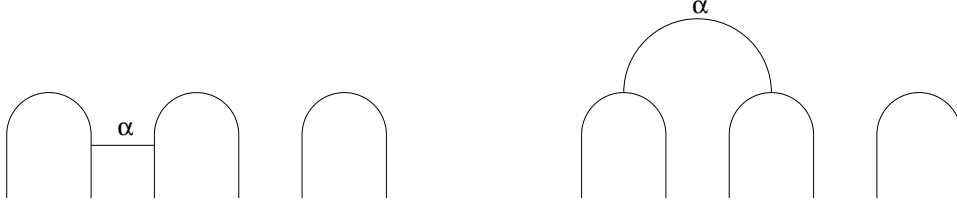


FIGURE 15. Relative bridge counts

always choose a level arc position for  $\tau$  in which the number of bridges of  $K_\tau$  equals  $\text{br}(K_\tau)$ .

**Definition 8.2.** Fix a level arc position for  $\tau$ . We refer to the number of bridges of  $K_\tau$  as the *bridge count* of  $K_\tau$  for this position, and denote it by  $\text{bc}(K_\tau)$ . If  $\alpha$  were moved from level arc position to the position shown in the right-hand diagram of figure 15, each of  $K_\lambda$  and  $K_\rho$  would contain a certain number of local maximum points, with the local maximum that lies on  $\alpha$  shared by both. We call those numbers the *relative bridge counts* of  $K_\lambda$  and  $K_\rho$  for the level arc position of  $\alpha$ , and denote them by  $\text{rbc}(K_\lambda)$  and  $\text{rbc}(K_\rho)$ . Notice that  $\text{bc}(K_\tau) = \text{rbc}(K_\lambda) + \text{rbc}(K_\rho)$ .

**Proposition 8.3.** Suppose that  $\tau$  is in a level arc position, and that a cabling operation as in figure 14 is performed, producing a new tunnel  $\tau'$  with principal vertex  $\{\rho, \tau, \tau'\}$ , and producing a tunnel arc  $\sigma$  for which  $\tau'$  is in a level arc position. In particular,  $K_{\tau'} \cup \sigma$  contains knots  $K'_\rho$  and  $K'_\tau$  equivalent to  $K_\rho$  and  $K_\tau$ . Then

- (1)  $\text{rbc}(K'_\rho) = \text{rbc}(K_\rho)$ .
- (2)  $\text{rbc}(K'_\tau) = \text{bc}(K_\tau)$ .
- (3)  $\text{bc}(K_{\tau'}) = \text{bc}(K_\tau) + \text{rbc}(K_\rho)$ .

*Proof.* Careful examination of figure 14 verifies the proposition in case the arc of the original  $K_\tau \cup \alpha$  dual to  $\lambda$  has one end that leaves  $\alpha$  in the upward direction and one end that leaves it in the downward direction. There are two other possibilities, either both ends leave in the upward direction, or both leave in the downward direction. Very similar constructions verify the proposition in those two cases.  $\square$

The next two results follow easily from proposition 8.3.

**Theorem 8.4.** Let  $\tau$  be a regular tunnel. In the principal path of  $\tau$ , in which  $\tau = \tau_n$ , let  $\tau_m$  be the first tunnel of depth 2, with principal vertex  $\{\tau_{m-2}, \tau_{m-1}, \tau_m\}$ . Put  $\tau_m$  in a level arc position, and let  $u_{m-2} = \text{rbc}(K_{\tau_{m-2}})$  and  $u_{m-1} = \text{rbc}(K_{\tau_{m-1}})$ . To the vertices  $\tau_m, \dots, \tau_n$ , assign values  $u_k$  inductively by the rule  $u_k = u_i + u_j$ , where the principal vertex of  $\tau_k$  is  $\{\tau_i, \tau_j, \tau_k\}$ . Then the bridge number of  $K_\tau$  is at most  $u_n$ .

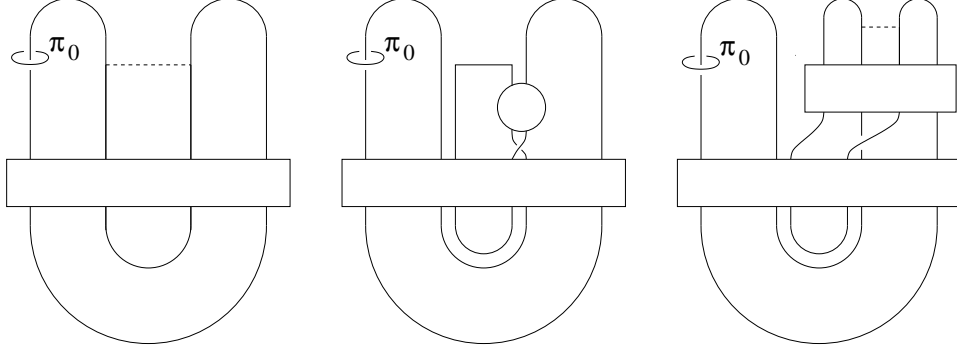


FIGURE 16. Conservative cabling of semisimple tunnels

**Corollary 8.5.** *Let  $\tau$  be a regular tunnel. In the principal path of  $\tau$ , in which  $\tau = \tau_n$ , let  $\tau_m$  be the first tunnel of depth 2, with principal vertex  $\{\tau_{m-2}, \tau_{m-1}, \tau_m\}$ . Suppose that the tunnel  $\tau_m$  can be put in a level arc position so that  $\text{rbc}(K_{\tau_j}) = \text{br}(K_{\tau_j})$  for  $m-2 \leq j \leq m-1$ , and  $\text{bc}(K_{\tau_m}) = \text{br}(K_{\tau_m})$ . Then the bridge number of  $K_\tau$  equals the value  $c_n$  of theorem 7.1.*

We now focus on cablings that produce semisimple tunnels. When Proposition 8.3 is applied at each step of the cabling sequence of a semisimple tunnel, each cabling construction can be performed so that the bridge number of the resulting  $K_\tau$  is 1 larger than the bridge number of the previous knot. Figure 16 (which, as in the proof of proposition 8.3, admits two variants) illustrates the inductive process. Each rectangle in that figure represents a pure 4-strand braid, and the circle represents a rational tangle. The first diagram shows a level simple tunnel of a 2-bridge knot  $K_{\tau_0}$ . The next cabling operation is performed, producing a knot  $K_{\tau_1}$  with a semisimple tunnel, as in the second diagram. This is moved by isotopy to the position in the third diagram; the tunnel is in a level arc position, and  $K_{\pi_0}$  has only one (relative) bridge. The  $k^{\text{th}}$  repetition of this construction produces a level arc position of  $\tau_k$  for which  $\text{bc}(K_{\tau_k}) = k+2$ ,  $\text{rbc}(K_{\tau_{k-1}}) = k+1$ , and  $\text{rbc}(K_{\pi_0}) = 1$ . Therefore we have:

**Theorem 8.6.** *Let  $\tau$  be a semisimple tunnel produced by  $m$  cabling constructions, and let  $\{\pi_0, \rho, \tau\}$  be its principal vertex. Then  $\tau$  can be placed in level arc position so that  $\text{bc}(K_\tau) = m+1$ ,  $\text{rbc}(K_\rho) = m$ , and  $\text{rbc}(K_{\pi_0}) = 1$ .*

**Corollary 8.7.** *Suppose that a semisimple tunnel  $\tau$  is produced by  $m$  cabling constructions. The  $\text{br}(K_\tau) \leq m+1$ .*

Combining theorem 8.6 with theorem 8.4, we have an upper bound for bridge number:

**Theorem 8.8.** *Let  $\tau$  be a regular tunnel. In the principal path of  $\tau$ , in which  $\tau = \tau_n$ , let  $\tau_m$  be the first tunnel of depth 2. Put  $u_{m-2} = m$  and  $u_{m-1} = m+1$ . To the vertices  $\tau_m, \dots, \tau_n$ , assign values  $u_k$  inductively by*



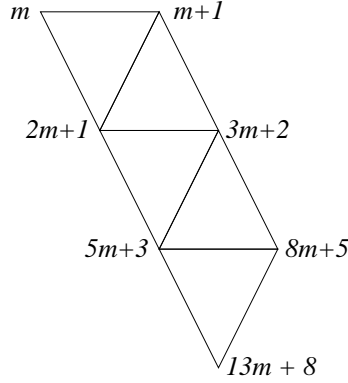


FIGURE 17. The fastest growing upper bounds for bridge number, starting with the last two semisimple tunnels in the cabling sequence.

the rule  $u_k = u_i + u_j$ , where the principal vertex of  $\tau_k$  is  $\{\tau_i, \tau_j, \tau_k\}$ . Then the bridge number of  $K_{\tau_n}$  is at most  $u_n$ .

We can give a universal upper bound for the bridge number of  $K_\tau$  in terms of the number of cablings that produce  $\tau$ . We denote by  $(F_1, F_2, \dots)$  the Fibonacci sequence  $(1, 1, 2, 3, \dots)$ .

**Theorem 8.9.** *Let  $\tau$  be a regular tunnel produced by  $n$  cabling operations, of which the first  $m$  produce semisimple tunnels. Then  $\text{br}(K_{\tau_n}) \leq mF_{n-m+2} + F_{n-m+1}$ .*

*Proof.* Figure 17 shows the type of principal path for which the  $u_k$  in theorem 8.8 grow most rapidly (in figure 17, the top two vertices are  $\tau_{m-2}$  and  $\tau_{m-1}$ , the last two semisimple tunnels that appear in the cabling sequence of  $\tau_n$ ). Putting  $u_{m-2} = m$ ,  $u_{m-1} = m+1$ , and  $u_k = u_{k-1} + u_{k-2}$  for  $k \geq m$ , theorem 8.8 gives  $\text{br}(K_{\tau_n}) \leq u_n$ . Since  $u_{m-2} = m \cdot F_1$  and  $u_{m-1} = m \cdot F_2 + F_1$ , the  $n - m$  additional recursions give the estimate in the theorem.  $\square$

For a fixed  $n$ , the largest upper bound in theorem 8.9 occurs when  $m = 2$ . We finish by showing that theorem 8.9 is sharp for that case.

**Theorem 8.10.** *The maximum bridge number of any tunnel number 1-knot having a tunnel produced by  $n$  cabling operations is  $F_{n+2}$ .*

*Proof.* Since the minimum possible value for  $m$  in theorem 8.9 is 2, any tunnel  $\tau$  produced by  $n$  cabling operations has  $\text{br}(K_\tau) \leq 2F_n + F_{n-1} = F_{n+2}$ . Now, let  $\tau_0$  be any simple tunnel. In [4], the slope sequences for the semisimple tunnels of 2-bridge knots were calculated, in particular finding that each cabling had slope of the form  $\pm 2 + 1/k$  for some integer  $k$ . Perform any cabling on  $K_{\tau_0}$  whose slope is not of this form, to produce a tunnel  $\tau_1$  for which  $K_{\tau_1} = 3$ . We now perform cabling constructions following the principal path indicated in figure 17 with  $m = 2$ . Theorem 7.1 shows that after  $n$  cabling constructions,  $\text{br}(K_\tau) \geq F_{n+2}$ .  $\square$

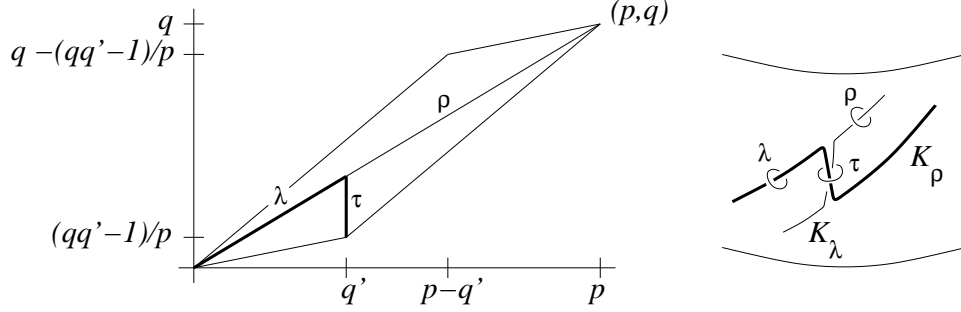


FIGURE 18. The properties of  $q'$ . The darker segments correspond to  $K_\rho$ , a  $(q', (qq' - 1)/p)$  torus knot. The picture on the right shows  $K_\lambda$  in the torus  $T \subset S^3$ , and  $K_\rho$  pulled slightly outside of  $T$ .

## 9. TUNNELS OF TORUS KNOTS

The tunnels of torus knots were analyzed by M. Boileau, M. Rost, and H. Zieschang [2]. There are two  $(1,1)$ -tunnels, and a third “short” tunnel represented by an arc that cuts straight across the complementary annulus when the knot is regarded as being contained in a standard torus. In certain cases, some of these tunnels are equivalent. In this section, we will analyze the cabling sequences for the short tunnels. In particular, we will see that their depths are arbitrarily large. On the other hand, all have distance 2, as we will now verify, while setting some notation for this section.

Consider a (nontrivial)  $(p, q)$  torus knot, contained in a standard torus  $T$  in  $S^3$ , bounding a solid torus  $W \subset \mathbb{R}^3 \subset S^3$ . The short tunnel is represented by an arc in  $T$ . Let  $H$  be a regular neighborhood of the knot and tunnel arc, chosen to intersect  $T$  in a regular neighborhood of the knot and tunnel arc in  $T$ . Let  $\tau$  be the cocore disk of the short tunnel, so that the torus knot is  $K_\tau$ . Now  $K_\tau$  is isotopic to a loop  $C$  in  $\partial H$  that lies entirely outside of  $W$  and is disjoint from  $\partial\tau$ . But  $C$  is also disjoint from the disk  $\overline{S^3 - H} \cap T$ , showing that  $\tau$  has distance 2.

We now begin our calculation of the slope invariants of the short tunnels. First we examine a cabling operation that takes a short tunnel  $\tau$  and produces a short tunnel of a new torus knot.

Assume for now that both  $p$  and  $q$  are positive. Since the  $(p, q)$  and  $(q, p)$  torus knots are isotopic, we may further assume that  $p > q$ . Let  $q'$  be the integer with  $0 < q' < p$  such that  $qq' \equiv 1 \pmod{p}$ . Figure 18 illustrates the features of  $q'$ . If the principal pair  $\{\lambda, \rho\}$  of  $\tau$  is positioned as shown in figure 18 (our inductive construction of these tunnels will show that the pair shown in the figure is indeed the principal pair), then  $K_\rho$  is a  $(q', (qq' - 1)/p)$  torus knot, and  $K_\lambda$  is a  $(p - q', q - (qq' - 1)/p)$  torus knot. We set  $(p_1, q_1) = (q', (qq' - 1)/p)$  and  $(p_2, q_2) = (p - q', q - (qq' - 1)/p)$ , so that  $K_\rho$  and  $K_\lambda$  are respectively the  $(p_1, q_1)$  and  $(p_2, q_2)$  torus knots.

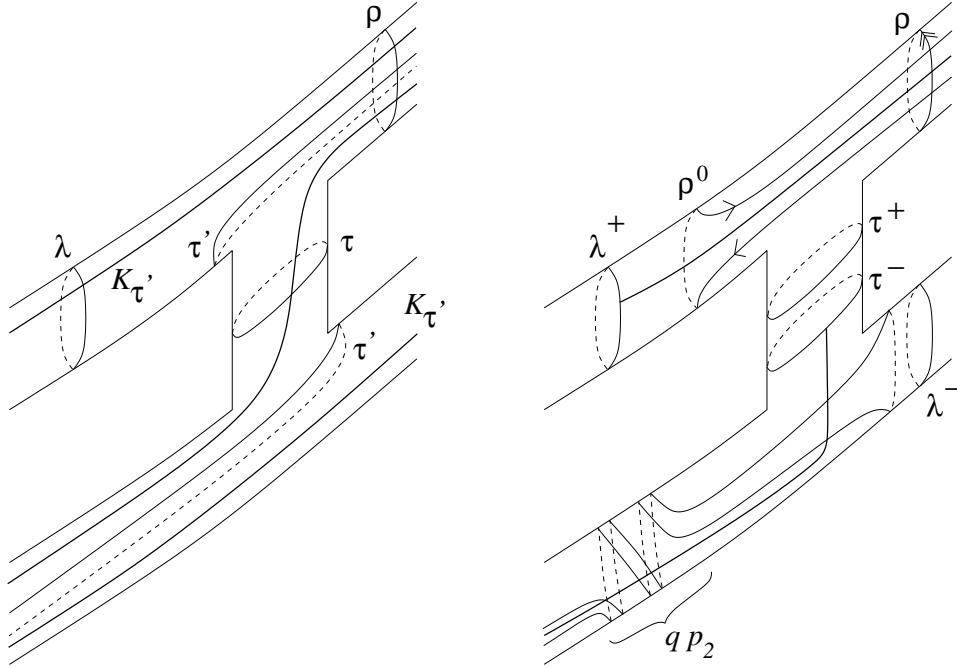


FIGURE 19. The cabling construction that replaces  $\rho$  (compare with figure 4). The left drawing shows the new tunnel disk  $\tau'$  and the knot  $K_{\tau'}$ . The right drawing shows a cabling arc for  $\tau'$ , running from  $\lambda^+$  to  $\tau^-$ , and the disks  $\rho$  and  $\rho^0$  used to calculate its slope. The  $qp_2$  turns of  $\rho^0$ , with the case  $qp_2 = 2$  drawn in the figure, make the copies of  $K_{\tau}$  and  $K_{\lambda}$  in its complement have linking number 0.

In figure 18, the linking number of  $K_{\rho}$  with  $K_{\lambda}$ , up to sign conventions, is  $q_1 p_2$ . One way to see this is to note that a Seifert surface for  $K_{\lambda}$  can be constructed from  $q_2$  meridian disks for  $W$  and  $p_2$  meridian disks for the “outside” solid torus  $S^3 - \overline{W}$ . When  $K_{\rho}$  is pulled slightly outside of  $W$ , as indicated in figure 18, each of the  $p_2$  meridian disks for the outside solid torus has  $q_1$  intersections with  $K_{\rho}$ , all crossing the disks in the same direction.

Figure 19 shows the new tunnel disk  $\tau'$  for a cabling construction that produces a  $(p+p_2, q+q_2)$ -torus knot  $K_{\tau'}$ . This disk meets  $T$  perpendicularly. The drawing on the right in figure 19 illustrates the setup for the calculation of the  $(\{\lambda, \tau\}; \rho)$ -slope pair of  $\tau'$ . Examination of that drawing shows that the slope pair of  $\tau'$  is  $[1, 2qp_2 + 1]$ .

As usual, let  $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ . If  $K_1$  is a  $(p_1, q_1)$ -torus knot and  $K_2$  is a  $(p_2, q_2)$ -torus knot, we denote by  $M(K_1, K_2)$  the matrix  $\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$ . In our case, this is the matrix  $M(K_{\rho}, K_{\lambda})$ . Adding the rows of

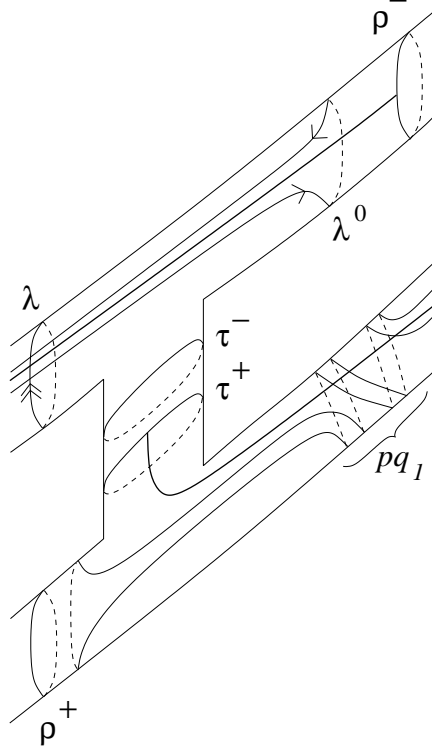


FIGURE 20. Calculation of the slope of  $\tau'$  for a cabling construction replacing  $\lambda$ . The cabling arc runs from  $\rho^-$  to  $\tau^+$ .

$M(K_\rho, K_\lambda)$  gives  $(p, q)$ , corresponding to  $K_\tau$ , so

$$M(K_\tau, K_\lambda) = U \cdot M(K_\rho, K_\lambda) .$$

The left drawing of figure 19 can be repositioned by isotopy so that  $\lambda$ ,  $\tau$ , and  $\tau'$  look respectively as did  $\lambda$ ,  $\rho$ , and  $\tau$  in the original picture, with  $\tau'$  as the tunnel of the  $(p + p_2, q + q_2)$  torus knot. Thus the procedure can be repeated, each time multiplying the matrix by another factor of  $U$ .

Figure 20 shows the calculation of the slope of the cabling construction replacing  $\lambda$  by a new tunnel  $\tau'$ , with the effect that

$$M(K_\rho, K_\tau) = L \cdot M(K_\rho, K_\lambda) .$$

Its slope pair works out to be  $[1, 2qp_1 - 1]$ . One might expect  $2qp_1 + 1$  as the second term, in analogy with the construction replacing  $\rho$ . However, as seen in figure 20, the  $pq_1$  twists needed in  $\lambda^0$  are in the same direction as the twists in the calculation for  $\rho$ , not in the mirror-image sense. This results in two fewer crossings of the cabling arc for  $\tau'$  with  $\lambda^0$  than before. In fact, the slope pairs for the two constructions can be described in a uniform way: For either of the matrices  $M(K_\tau, K_\lambda)$  and  $M(K_\rho, K_\tau)$ , a little bit of arithmetic shows that the second entry of the slope pair for the cabling operation that

produced them is the sum of the product of the diagonal entries and the product of the off-diagonal entries, that is,  $[1, pq_2 + qp_2]$  in the first case and  $[1, pq_1 + qp_1]$  in the second.

We can now describe the complete cabling sequence. Still assuming that  $p$  and  $q$  are both positive and  $p > q$ , write  $p/q$  as  $[n_1, n_2, \dots, n_k]$  with all  $n_i$  positive. We may assume that  $n_k \neq 1$ . According as  $k$  is even or odd, consider the product  $U^{n_k} L^{n_{k-1}} \dots U^{n_2} L^{n_1}$  or  $L^{n_k} U^{n_{k-1}} \dots U^{n_2} L^{n_1}$ . Start with a trivial knot regarded as a  $(1, 1)$ -torus knot, and a tunnel positioned so that  $K_\rho$  is a  $(1, 0)$ -torus knot and  $K_\lambda$  is a  $(0, 1)$ -torus knot. The corresponding matrix  $M(K_\rho, K_\lambda)$  is the identity matrix. Multiplying by  $L^{n_1}$  has the effect of doing  $n_1$  trivial cabling constructions, each with slope 1, and ending with the trivial knot positioned as an  $(n_1 + 1, 1)$ -torus knot. At that stage,  $M(K_\rho, K_\lambda)$  is  $\begin{pmatrix} 1 & 0 \\ n_1 & 1 \end{pmatrix}$ . Then, multiplying by  $U$  corresponds to a true cabling construction. In the above notation, the new matrix  $M(K_\tau, K_\lambda)$  is  $\begin{pmatrix} n_1 + 1 & 1 \\ n_1 & 1 \end{pmatrix}$ , and the knot  $K_{\tau'}$  is a  $(2n_1 + 1, 2)$ -torus knot. As explained above, the construction has slope pair  $[1, 2n_1 + 1]$ , so the simple slope is  $m_0 = [1/(2n_1 + 1)]$ . Continue by multiplying  $n_2 - 1$  additional times by  $U$ , then  $n_3$  times by  $L$  and so on, performing additional cabling constructions with slopes calculated as above from the matrices of the current  $K_\rho$ ,  $K_\lambda$ , and  $K_\tau$ .

At the end, there is no cabling construction corresponding to the last factor  $L$  or  $U$ . For specificity, suppose  $k$  was even and the product was  $U^{n_k} L^{n_{k-1}} \dots U^{n_1} L^{n_1}$ . At the last stage, we apply  $n_k - 1$  cabling constructions corresponding to multiplications by  $U$ , and arrive at a tunnel  $\tau$  for which  $M(K_\rho, K_\lambda)$  is  $U^{n_k-1} L^{n_{k-1}} \dots U^{n_2} L^{n_1}$ . The sum of the rows is then  $(p, q)$  (multiplying by  $U$  and using the case “ $q/s$ ” of lemma 14.3 of [4]), so  $K_\tau$  is the  $(p, q)$  torus knot. The case when  $k$  is odd is similar (using the “ $p/r$ ” case of lemma 14.3 of [4] at the end). In summary, there are  $-1 + \sum_{i=2}^k n_i$  (nontrivial) cabling constructions, whose slopes can be calculated as above.

Suppose now that  $p$  is positive but  $q$  is negative. We may assume that  $p > |q|$ . We have already found the cabling sequence for the case of the  $(p, -q)$ -torus knot, and since reversing orientation negates the slope parameters in the Parameterization Theorem (remark 12.5 of [4]), we need only negate its slopes to obtain the cabling sequence for the  $(p, q)$ -torus knot.

**Remark 9.1.** Figure 21 shows an initial segment of the principal path for the tunnels of the  $(p, q)$ -torus knots for the continued fractions  $p/q = [1, 2, 2, \dots, 2]$ , which limit to  $\sqrt{2}$ . Notice that this is the path of cheapest descent from figure 11. The small numbers along the path are the slopes, the letters indicate whether the constructions correspond to multiplication by  $U$  or by  $L$ , and the pairs show the  $(p, q)$  for the torus knots determined by the tunnel at each step. The first nontrivial cabling, with  $m_0 = [1/3]$ , produces a  $(3, 2)$ -torus knot, and the second produces a  $(4, 3)$ -torus knot with bridge

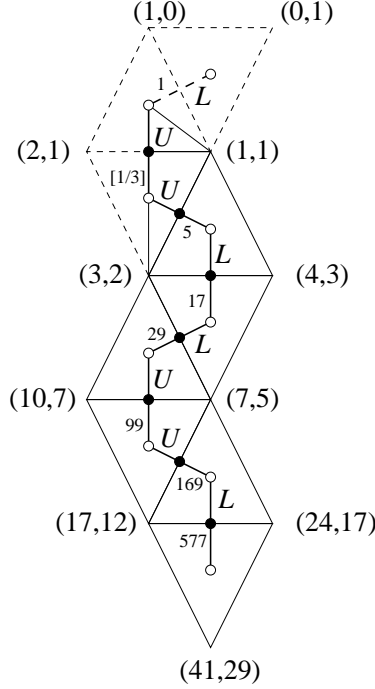


FIGURE 21. Slowest growth of bridge number as a function of depth for torus knot tunnels, corresponding to the continued fraction expansion  $41/29 = [1, 2, 2, 2, 2]$ . The  $(41, 29)$  torus knot has the smallest bridge number of any torus knot with a depth 4 tunnel.

number 3. Since we always have  $p > q$ , the bridge number is simply the value of  $q$ . These obey the recursion of corollary 7.4, starting with  $b_2 = 2$  and  $b_3 = 3$ . Since the cabling sequence for the short tunnel  $\tau$  of any torus knot contains only one two-bridge knot (the  $(2n_1 + 1, 2)$ -torus knot produced by the first nontrivial cabling), there is no regular torus knot tunnel which has  $b_3 = 2$ . Therefore each  $b_{2d}$  in this sequence (which also occur for its reversed-orientation sequence, where the negatives of these slopes are used) gives the minimum bridge number for a torus knot with a tunnel of depth  $d$ . Finally, we note that  $b_{2d} = a_d$  where  $a_1 = 2$ ,  $a_2 = 5$ , and  $a_d = 2a_{d-1} + a_{d-2}$  for  $d \geq 3$ . The asymptotic growth rate of this sequence is proportional to  $(1 + \sqrt{2})^d$ , which as we have seen is the minimum rate in general.

**Remark 9.2.** Considering the preceding example, we can see how to determine the depth from the continued fraction expansion  $[n_1, n_2, \dots, n_k]$  of  $p/q$ . Ignore  $n_1$ , since it corresponds to cablings which produce the trivial tunnel. Basically, each of the remaining  $n_i$  increases the depth by 1. However, for a block  $[n_i, \dots, n_{i+j}]$  with each  $n_\ell = 1$  for  $i \leq \ell \leq i+j$ , and  $n_{i-1} \neq 1$  and  $n_{i+j+1} \neq 1$ , only the cablings corresponding  $n_i$ ,  $n_{i+2}$ ,  $n_{i+4}$ , and so on

increase the depth (consider the principal path drawn as in figure 8). Also, a final block  $[n_{k-1}, n_k] = [1, 2]$  increases the depth by only 1, since the final 2 represents only a single cabling operation.

The exceptional cases of M. Boileau, M. Rost, and H. Zieschang [2], that is, the cases when there are fewer than three tunnels, are exactly the cases when  $\tau$  is semisimple. To understand this computationally, we may assume that  $p > q \geq 2$ , and we find:

*Case I:*  $p \equiv 1 \pmod{q}$ . We have  $p/q = [n_1, q]$  and we examine  $U^q L^{n_1}$ . There are  $n_1$  trivial cablings, producing the trivial tunnel, then there are  $q - 1$  cabling constructions retaining one of the original arcs of the trivial knot, showing that  $\tau$  is semisimple.

*Case II:*  $p \equiv -1 \pmod{q}$ .

In these cases,  $p/q$  is  $[n_1, 1, q]$ , with  $q > 1$ . Examining  $L^{q-1}UL^n$ , the first nontrivial cabling corresponds to  $U$ , and produces a simple tunnel, then the  $q - 1$  cablings corresponding to the  $L^q$  retain one of the original arcs of  $K_{\tau_0}$ . Thus these are also semisimple tunnels.

In all other cases, the continued fraction expansion of  $p/q$  either has more than three terms, or has second term greater than 1, so the regular tunnel is regular. We have verified the equivalence of the first two conditions in the following proposition. As already noted, the equivalence of the first and third is from [2].

**Proposition 9.3.** *For the  $(p, q)$  torus knot  $K_{(p,q)}$ , every tunnel has distance 2. The following are equivalent:*

- (1)  $p \not\equiv \pm 1 \pmod{q}$ .
- (2) *The short tunnel is regular.*
- (3)  $K_{(p,q)}$  *has exactly three tunnels.*

We have implemented the algorithms for the slope sequence and the depth of the short tunnel computationally [5]. Some sample calculations are:

```
TorusKnots> slopes 41 29
[ 1/3 ], 5, 17, 29, 99, 169, 577

TorusKnots> slopes 181 (-48)
[ 6/7 ], -15, -23, -31, -151, -271, -883, -2157, -3431

TorusKnots> depth 41 29
4

TorusKnots> [ depth 41 n | n <- [2..40] ]
[1,1,1,1,1,1,1,2,3,2,1,2,3,3,3,2,1,1,2,3,3,3,3,2,3,4,3,2,3,
2,2,2,2,2,2,2,1]
```

The last command produces a list of the depths of the short tunnels for the torus knots  $K_{(41,2)}$  through  $K_{(41,40)}$ .

## 10. THE CASE OF TUNNEL NUMBER 1 LINKS

As explained in [4], our entire theory can be adapted to include tunnels of tunnel number 1 links simply by adding the separating disks as possible slope disks. The full disk complex  $\mathcal{K}(H)$  is only slightly more complicated than  $\mathcal{D}(H)$ . Each separating disk is disjoint from only two other disks, both nonseparating, so the additional vertices appear in 2-simplices attached to  $\mathcal{D}(H)$  along the edge opposite the vertex that is a separating disk. The quotient  $\mathcal{K}(H)/\mathcal{G}$  only has such additional 2-simplices: (1) there is a unique orbit of “primitive” separating disk, consisting of separating disks disjoint from a primitive pair, which are exactly the intersections of splitting spheres with  $H$ . Their orbit  $\sigma_0$  is a vertex of a “half-simplex”  $\langle \sigma_0, \pi_0, \mu_0 \rangle$  attached to  $\mathcal{D}(H)/\mathcal{G}$  along  $\langle \pi_0, \mu_0 \rangle$ . It is the unique tunnel of the trivial 2-component link, and has simple slope  $\infty$ . (2) Simple separating disks lie in half-simplices attached along  $\langle \pi_0, \mu_0 \rangle$ , just like nonseparating simple disks. Their simple slopes are  $[p/q]$  with  $q$  even. (3) the remaining separating disks lie in 2-simplices attached along edges of  $\mathcal{D}(H)/\mathcal{G}$  spanned by two (orbits of) disks, at least one of which is nonprimitive. A single “Y” is added to  $\mathcal{T}$  for each added 2-simplex (or a folded “Y”, for the half-simplices). The link in  $\mathcal{K}'(H)/\mathcal{G}$  of a link tunnel is simply the top edges (or top edge, for the trivial and simple tunnels) of such a “Y”.

Cabling operations differ only in allowing separating slope disks, which produce a tunnel of a tunnel number 1 link. The cabling sequence ends with the first separating slope disk, and cannot be continued. The Parameterization Theorem holds as stated, except allowing  $q_n$  to be even, and allowing  $m_0 = \infty$  for the unique tunnel of the trivial link.

For link tunnels, the distance and depth invariants are defined as for knot tunnels. Depth 1 tunnels are the tunnels of links with one component unknotted. The other component must be a  $(1,1)$ -knot, and the link must have torus bridge number 2 [4]. Lemma 4.1 holds when  $\tau$  is separating, in fact the argument is an easier version of the argument in [7], so theorem 4.3 and corollary 4.5 hold for links as well as knots.

For a tunnel  $\sigma$  of a tunnel number 1 knot, there is a version of a giant step that produces a tunnel number 1 link. Choose any loop  $L_1$  in  $\partial H$  that crosses  $\sigma$  exactly once, and let  $\sigma'$  be the frontier of a regular neighborhood of  $L_1 \cup \sigma$  in  $H$ . Since  $\sigma'$  is separating, the core circles of its complementary solid tori form a tunnel number 1 link with tunnel  $\sigma'$ ; one of these core circles is isotopic to  $L_1$ . One might even describe a giant step starting from a tunnel of a link, but this is of little interest since such a giant step could not appear in a minimal giant step sequence starting from  $\pi_0$ , because the two disks disjoint from a separating disk are also disjoint from each other. Section 6 adapts almost word-for-word to allow tunnels of links.

The proof of theorem 7.1 adapts without difficulty to the case of links since the tunnel leveling of [6] applies to links as well as knots. Since the geometric



constructions in sections 7 and 8 also work for cabling constructions that produce links, the results of both those sections apply just as well to links.

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